# BASIC NUMERICAL SKILLS

**III Semester**

**COMMON COURSE**

## B Com/BBA

*(2011 Admission)*


## UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Calicut University P.O. Malappuram, Kerala, India 673 635


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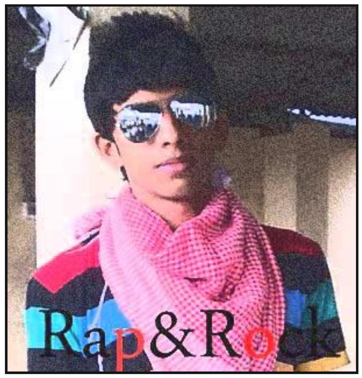


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**STUDY MATERIAL**

**Common Course for**

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# BASIC NUMERICAL SKILLS

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# MODULE - I

The theory of sets was introduced by the German mathematician Georg Cantor in 1870. A set is well defined collection of distinct objects. The term well defined we mean that there exists a rule with the help of which we will be able to say whether a particular object 'belong to' the set or does not belong to the set. The objects in a set are called its members or elements.

The sets are usually denoted by the Capital letters of the English alphabet and the elements are denoted by small letters.

If x is an element of a set A, we write X∈A (read as x belong to A). If x is not an element of A then we write X∉A (read as x does not belong to A).

**Representation of a Set or Methods of describing a Set**

A set is often representation in two ways:

(1) Roster method or tabular or enumeration method.

(2) Set builder method or Rule or Selector method.

**Tabular Method**

In this method, a set is described by listing the elements, separated by commas and are enclosed within braces. For example the set of first three odd numbers 1,3,5 is represented as :

A = {1, 3, 5}

**Set Builder Method**

In this method, the set is represented by specifying the characteristic property of its elements. For example the set of natural numbers between 1 and 25 is represented as:

A = $\{x : x \in N \text{ and } 1 < x < 25 \}$

**TYPES OF SETS**

**1. Null Set or Empty Set or Void Set**

A set containing no element is called a null set. It is denoted by { } or ∅

Eg:- the set of natural numbers between 4 and 5.

**2. Singleton or Unit Set**

A Set containing a single element is called singleton set

Eg:- Set of all positive integers less than 2

**3. Finite Set**

A Set is said to be a finite set if it consist only a finite number of elements. The null set is regarded as a finite set.

Eg:- the set of natural numbers less than 10

**4. Infinite Set**

A set is said to be an infinite set if it consists of a infinite number of elements.

Eg:- Set of natural numbers.

**5. Equvilant Set**

Two sets A and B are said to be equivalent set if they contain the same number of elements

Eg:- Let A = {1, 2, 3 } and B = {$a$, $b$, $c$ }

**6. Equal Set**

Two sets A and B are said to be equal if they contain the same elements.

Eg:- Let A = {1, 2, 3 } B = {2, 1, 3 }

**7. Sub Set and Super Set**

If every element of A is an element of B then A is called a subset of B and symbolically we write A⊆B

If A is contain in B then B is called super set of A and written as B ⊇A

Eg: A = {2, 3 } and B = {2, 3, 4 } then A is a proper subset of B

**8. Power Set :-**

The collection of all sub sets of a set A is called the power set of A. It is denoted by P(A). In P(A), every element is a set. For example

A = {1, 2, 3 }

Then P(A) ={ }, {1, 2, 3 } {1}{ 2, } {3 }{1, 2 } {1, 3 } {2, 3 }

**9. Universal Set**

If all the sets under consideration are subsets of a fixed set U, is called universal set. For example A is the set of vowels in the English Alphabet. Then the set of all letters of the English Alphabet may be taken as the universal set.

**10. Disjoint Set**

Two sets A and B are said to be disjoint sets if no element of A is in B and no element of B is in A. For example

A= {3, 4, 5 }, B = {6, 7, 8 }

## SET OPERATIONS

**(1) Union of sets :**

The union of two sets A and B is the set of all those elements which belongs to A or to B or to both. We use the notation A∪B to denote the union of A and B.

For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A∪B = {1, 2, 3, 4, 5, 6 }

**(2) Intersection of Sets**

The intersection of two sets is the set consisting of all elements which belong to both A and B. It is denoted by A∩B. For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A∩B = {3, 4 }

**(3) Difference of two sets**

The difference of the two sets A and B is the set of all elements in A which are not in B. It is denoted by A-B or A/B. For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A—B = {1, 2 }

**(4) Complement of a set**

Complement of a set is the set of all element belonging to the universal set but not belonging to A. It is denoted by $A^c$ or $A'$

$A^c$= ∪-A. For example If ∪= {1, 2, 3, 4,5 } A = {1,3, 5} , Then $A^c$ = { 2, 4 }

**ALGEBRA OF SETS OR LAWS OF SET OPERATION**

**(1) Commutative Laws :-**

If A and B are any two sets then :-

(i)A∪B =B∪A

(ii)A∩B = B∩A

**(2) Associative Laws**

If A, B and C are three sets, then

(i)A∪ (B∪C) = (A∪B) ∪C and

(ii)A∩ (B∩C) = (A∩B) ∩C

**Distributive Laws**

If A, B, C are any three sets, then

(i) A∪ (B∩C) = (A∪B) ∩(A∪C) and

(ii) A∩ (B∪C) = (A∩B) ∪ (A ∩C)

De-Morgan's Law

If A and B are any two subsets of '∪', then

(i)(A∪B)' =A'∩B'

That is complement of union of two sets equal to the intersection of their complements.

(ii) (A∩B)' =A'∪B'

That is complement of intersection of two sets is equal to the union of their complements.

Practical Problems

1) If A = {1, 2, 3, 4 }, B = {3, 4,5,6 }

C ={5, 6, 7, 8 } D = {7, 8, 9, 10 }

Find (i) A∪B (ii) A∪C (iii) B∪C

(iv) B∪D (v) A∪B∪C (vi)A∪B∪D (vii)B∪C∪D

**Answer**

(i) A∪B = {1, 2, 3, 4, 5, 6 }

(ii) A∪C = {1, 2, 3, 4, 5, 6, 7,8 }

(iii) B∪C ={3, 4,5, 6, 7, 8 }

(iv) B∪D = {3, 4, 5, 6, 7, 8, 9, 10}

(v) A∪B∪C ={1, 2, 3, 4, 5, 6, 7, 8 }

(vi) A∪B∪D = {1, 2, 3, 4,5, 6, 7, 8, 9, 10 }

(vii) B∪C∪D = {3, 4,5, 6, 7, 8, 9, 10 }

2) If A ={1, 3, 5, 7 }, B = { 5, 9, 13, 17 } C = {1, 3, 9, 13 }

Find (i) A∩B (ii) B∩A (iii) A-B (iv) B—A (v) A—C (vi)(A—B) —C (vii) A—(A-B)

**Answer**

(i) A∩B = {5}

(ii) B∩A = { 5 }

(iii) A-B ={1,3, 7 }

(iv) B-A = {9, 13, 17}

(v) A—C ={5, 7}

(vi) (A—B) —C) = {7 }

(vii) A-(A-B) = {5 }

3) A = $\{x : x \ is \ a \ natural \ number \ satisfy \ 1 < x \ \leq 6\}$

B = $\{x : x \ is \ a \ natural \ number \ satisfy \ 6 < x \ \leq 10\}$

Find (i) A∪B (ii)A∩B

**Answer**

A = { 2, 3, 4, 5, 6 }

B = {7,8, 9, 10 }

(i) A∪B = {2, 3, 4, 5, 6, 7,8, 9, 10 }

(ii) A∩B = { }

4) Let ∪ = { 1,2, 3, 4, 5, 6, 7, 8, 9,10 } and A = {1, 3, 5, 7, 9 } Find $A^C$.

**Answer**

$A^C$ means belongs to universe but not in A

$A^C$ = {2, 4, 6, 8, 10}

5) Let ∪= { 1,2, 3, 4, 5, 6, 7, 8, 9,10 }

A = { 1, 4, 7,10 } B = { 2, 4, 5, 8 }

Find A'∩B

**Answer:**

A' = Belongs to universe but not in A

A' = { 2, 3, 5, 6, 8, 9 }

A'∩B = { 2, 5, 8 }

=====

7) Let A = { 1,2,3 } B = { 2, 4, 5}

C = { 2, 4, 6}, ∪= { 1, 2, 3, 4, 5, 6, 7, 8 }

Verify that (i) (A∩B) = A'∩B'

(ii) (A∩B) = A'∪B'

**Answer**

(i) (A∪B)' = ∪ − (A∪B)

A∪B = { 1, 2, 3, 4, 5}

(A∪B)' = { 6, 7, 8}

A' = ∪ −A = { 4, 5, 6, 7, 8}

B' = ∪ −B = { 1,  3,  6, 7, 8}

A'∩B' = { 6, 7, 8}

Hence (A∪B)' = A'∩B'

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(ii) (A∩B)' = ∪ − (A∩B)

(A∩B) = { 2}

(A∩B)' = { 1, 3, 5, 6, 7, 8}

A'∪B' = { 1, 3, 5, 6, 7, 8}

Hence (A∩B)'= A'∪B'

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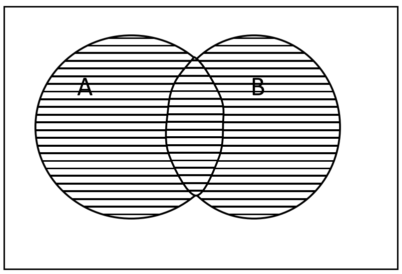
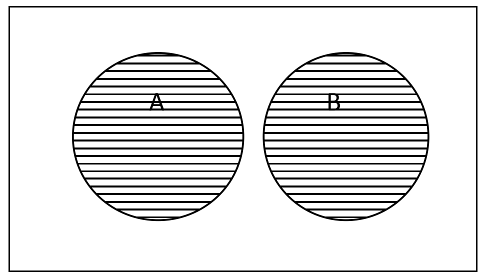
**VENN DIAGRAM**

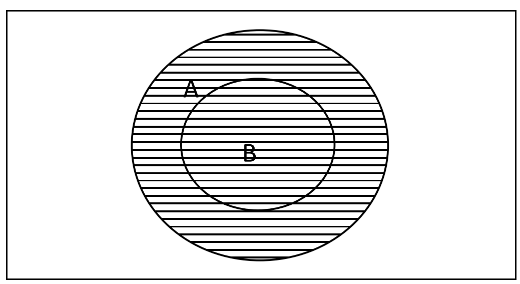
The relationship between sets can be represented by means of diagrams. It is known as Venn diagram. It consists of a rectangle and circles. Rectangle represents the universal set and circle represents any set.

For example A∪B, A∩B, A−B, and $A^C$ can be represented as follows:

**(1)A∪B**

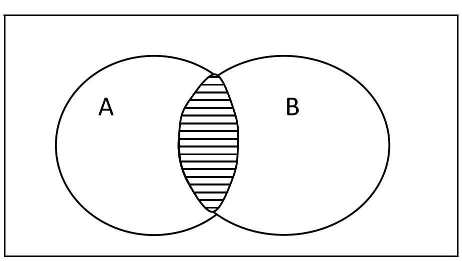


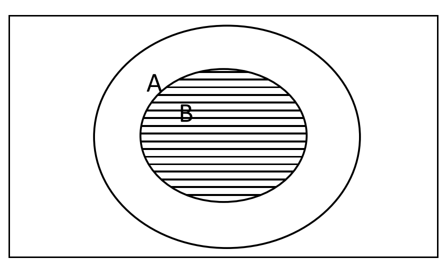



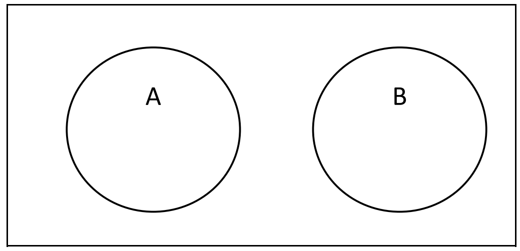




In diagram I A and B are intersecting in the second diagram, A and B are disjoint and in the third figure, B is a subset of A. In all the diagrams, A∪B is equal to the shaded area.

**(ii) A∩B**

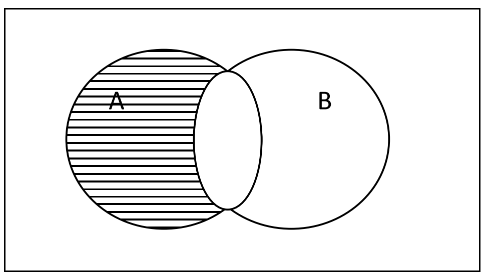


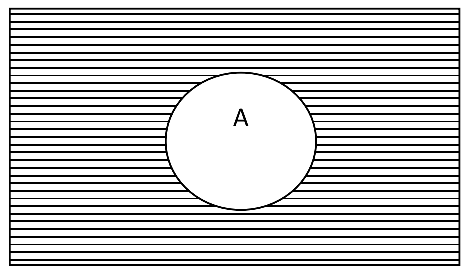



In first diagram A∩B is marked by lines. In the second diagram B is a subset of A and A∩B is also marked by lines. In the third diagram A and B are disjoint and therefore there is no intersection and so A∩B = ∅

**(iii)A−B** **(iv) $A^c$**




A – B ie belongs to A but not in B is shaded by lines




$A^c$ i.e. belongs to universe but not in A is shaded by lines

**Theorems on Number of Elements in a Set**

(i) n(A∪B) = n(A) + n(B) – n(A∩B)

(ii) n(A∪B∪C) = n(A) + n(B) + n(C) – n(A∩B) – n(A∩C) – n(B∩C) + n(A∩B∩C)

A∪B = at least one of them

A∩B = both A & B

A∪B∪C = At least one of them

A∩B∩C = All of them

1) Among 60 people, 35 can speak in English, 40 in Malayalam and 20 can speak in both the languages. Find the number of people who can speak at least one of the languages. How many cannot speak in any of these languages?

**Answer**

n(A) = Speak in English

n(B) = Speak in Malayalam

Given

$n(A) = 35$,   $n(B) = 40$

$n(A \cap B) = 20$

$A \cup B$ = (ie people who speak in at least one of the language) =

$n(A) + n(B) - n(A \cap B)$

$= 35 + 40 - 20 = 55$

Number of people who cannot speak in any one of these language = 60- 55 = 5

2) Each student in a class, studies at least one of the subject English, Mathematics and Accountancy. 16 study English, 22 Accountancy and 26 Mathematics. 5 study English and Accountancy, 14 study Mathematics and Accountancy and 2 English, Accountancy and Mathematics. Find the number of student who study

(i) English & Mathematics

(ii)English, Mathematics but not Accountancy

**Answer**

Let A = students study English

B = students study Mathematics

C = students study Accountancy

Given

$n(A\ ) = 16$, $n(B)\ 26$, $n(C)\ 22$

$n(A \cap C) = 5$, $n\ (B \cap C) = 14$, $n(A \cap B)$?

$n(A \cap B \cap C) = 2$, $n(A \cup B \cup C) = 40$

We know that

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$40 = 16 + 26 + 22 - n(A \cap B) - 5 - 14 + 12$

$n(A \cap B) = 16 + 26 + 22 - 5 - 14 + 2 - 40$

$= 7$

∴Number of students study for English & Mathematics = 7

Number of student who study English, Mathematics but not Accountancy = $n(A \cap B \cap C')$

$n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C)$

$= 7 - 2 = 5$

Number of student who study English, Mathematics and not Accountancy = 5

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3) In a college there are 20 teachers, who teach Accountancy or Statistics. Of these 12, teach Accountancy and 4 teach both Statistics and Accountancy. How many teach Statistics?

**Answer**

Let n(A) = teachers teach Accountancy

n(B) = teacher teach Statistics

Given n(A) = 12, n(B) ?

$n(A \cap B) = 4$, $n(A \cup B) = 20$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$20 = 12 + n(B) - 4$

$n(B) = 20 - 12 + 4 = 12$

Number of teachers teach Statistics = 12

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4) Out of 2400 students who appeared for BCom degree Examination, 1500 failed in Numerical skills, 1200 failed in Accountancy and 1200 failed in Informatics, 900 failed in both Numerical skills and Accountancy 800 failed in both Numerical skills and Informatics, 300 failed in Accountancy and Informatics, 40 failed in all subjects. How many students passed all three subjects?

**Answer**

Let A = number of students failed in Numerical Skills

B = number of students failed in Accountancy

C = number of students failed in Informatics

Given

$n(A) = 1500$, $n(B) = 1200$, $n(C) = 1200$

$n(A \cap B)$ 900, $n(A \cap C)$800, $n(B \cap C) = 300$

$n(A \cap B \cap C) = 40$

Number of students failed in at least one subject = $n(A \cup B \cup C)$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$= 1500 + 1200 + 1200 - 900 - 800 - 300 + 40 = 1940$

Number of student passed in all subjects = $2400 - 1940 = 460$

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# MATRICES

A matrix is an ordered rectangular array of numbers or functions. It is a rectangular presentation of numbers arranged systematically in rows and columns one number or functions are called the elements of the matrix. The horizontal lines of elements of the matrix are called rows and vertical lines of elements of matrix are called columns.

## Order Of Matrix

A matrix having 'm' rows 'n' columns are called a matrix of order 'm x n' or simply 'm x n' matrix (read as an 'm' by 'n' matrix)

Types of Matrices

(i) **Rectangular matrix** : Any matrix with 'm' rows and 'n' column is called a rectangular matrix. It is a matrix of Order m x n. For example,

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 \end{bmatrix}$ is a 3 x 4 matrix

(ii) **Square matrix :** A matrix by which the number of rows are equal to the number of columns, is said to be a square matrix. Thus an m x n matrix is said to be square matrix if m= n and is known as a square matrix of order 'n'. For example,

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ is a square matrix of order 3

(iii) **Row matrix** : A matrix having only one row is called a row matrix. For example,

$A = \begin{bmatrix} 1 & 2 & 3 & 2 \end{bmatrix}$ is a row matrix.

(iv) **Column matrix** : A matrix having only column is called column matrix. For example,

$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ is a column matrix.

(v) **Diagonal matrix** : A square matrix is said to be diagonal it all elements except leading diagonal are zero. Elements $a_{11}$, $a_{22}$, $a_{33}$ etc. termed as leading diagonal of a matrix. Example of Diagonal matrix is

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 9 \\ 0 & 0 & 6 \end{bmatrix}$ is a diagonal matrix. Leading diagonal elements are 2, 3, 6.

(vi) **Scalar Matrix** : A diagonal matrix is said to be scalar matrix, if its diagonal elements are equal. For example.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(vii) **Unit matrix of identity matrix** : A diagonal matrix in which diagonal elements are 1 and rest are zero is called Unit Matrix or identity matrix. It is denoted by 1.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a Unit matrix or Identity matrix.

(viii) **Null Matrix or Zero matrix:** A matrix is said to be zero or null matrix if all its elements are zero. For example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is a Null matrix or Zero matrix

(ix) **Triangular matrix:** If every element above or below the leading diagonal is zero, the matrix is called Triangular matrix. It may be upper triangular or lower triangular. In upper triangular all elements below the leading diagonal are zero and in the lower triangular all elements above the leading diagonal are zero. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$ is a matrix of upper triangular.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$ is matrix of lower triangular

(x) **Symmetric matrix :** Any square matrix is said to be symmetric if it is equal to transpose. That is, $A = A^t$

Transpose of a matrix as a matrix obtained by interchanging its rows and columns. It is denoted by $A^t$ or $A^1$. Example of symmetric matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

(xi) **Skew Symmetric Matrix** : Any square matrix is said to be skew symmetric if it is equal to its negative transpose. That is $A = {}^-A^t$

For example $$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = A^t$$

$$A^t = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$

$$-A^t = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

**Operation of matrices**

Operation of matrices relate to the addition of matrices, difference, multiplication of matrix by a scalar and multiplication of matrices.

Addition of matrices : If A and B are any two matrices of the same order, their sum is obtained by the elements of A with the corresponding elements of B.
For example :

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -7 & 3 & 2 \\ -4 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 & 3 \\ -3 & -2 & 1 \\ 3 & -2 & 2 \end{bmatrix}$$

$$\text{Then } A + B = \begin{bmatrix} 3 & -4 & 5 \\ -10 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Difference of Matrices : if A and B are, two matrices of the same order, then the difference is obtained by deducting the element of B from A.

$$\text{If} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\text{Then } A - B = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 3 & -1 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar
The elements of Matrix A is multiplied by any value (ie. K) and matrix obtained is denoted by K

$$\text{For example : } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\text{Then } 5A = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix}$$

**Practical Problems**

1) If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ - $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ Find 3A – B ?

Ans: $3A = \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix}$

$$3A - B = \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 6 \\ 5 & -1 & 7 \end{bmatrix}$$

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(2) Solve the equation:

$$2\begin{bmatrix} x & y \\ z & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 5\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

======

Ans : $2\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2z & 2t \end{bmatrix}$

$$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$$

$$5\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 \\ 20 & 30 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x & 2y \\ 2z & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 \\ 20 & 30 \end{bmatrix}$$

$$2x + 3 = 15, 2x = 15 - 3 = 12, x = \frac{12}{2} = 6$$

$$2y + -3 = 25, 2y = 25 + 3 = 28,\ y = \frac{28}{2} = 14$$

$$2z + 0 = 20, 2z = 20,\ z = \frac{20}{2} = 10$$

$$2t + 6 = 30,\ 2t = 30 - 6 = 24,\ t = \frac{24}{2} = 12$$

(3) Find the value of a, b if

$$2 \text{ x } \begin{bmatrix} a & 5 \\ 7 & b-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Ans : $2 \text{ x } \begin{bmatrix} a & 5 \\ 7 & b-3 \end{bmatrix} = \begin{bmatrix} 2a & 10 \\ 14 & 2b-6 \end{bmatrix}$

$$\begin{bmatrix} 2a & 10 \\ 14 & 2b-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

2a + 3 = 7, 2a = 7-3 = 4, a= $\frac{4}{2}$ = 2

2b – 6 + 2 = 14, 2b = 14 + 6 – 2 = 18, b = $\frac{18}{2}$ = 9

**<u>Multiplication of two matrices</u>**

For multiplication, take each row of the left hand side matrix with all colums of the right hand side matrix. For example A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ B = $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ Then AB = $\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

**Practical Problems**

(1) Let A = $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix}$ B= $\begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}$ Compute AB

Ans: AB = $\begin{bmatrix} 1x2 + 2x5 + 3x1 & 1x3 + 2x4 + 3x5 & 1x1 + 2x2 + 3x3 \\ -2x2 + 1x5 + 4x1 & -2x3 + 1x4 + 4x5 & -2x1 + 1x2 + 4x3 \end{bmatrix}$

AB = $\begin{bmatrix} 2 + 10 + 3 & 3 + 8 + 15 & 1 + 4 + 9 \\ -4 + 5 + 4 & -6 + 4 + 20 & -2 + 2 + 12 \end{bmatrix}$

AB = $\begin{bmatrix} 15 & 26 & 14 \\ 5 & 18 & 12 \end{bmatrix}$

(2) Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ Find AB and BA and hence show that AB≠ BA

Ans : AB = $\begin{bmatrix} 1x - 2 + 2x1 & 1x2 + 2x - 1 \\ 3x - 2 + 4x1 & 3x2 + 4x - 1 \end{bmatrix}$

$\begin{bmatrix} -2 + 2 & 2 + -2 \\ -6 + 4 & 6 + -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}$

BA = $\begin{bmatrix} -2x1 + 2x3 & -2x2 + 2x4 \\ 1x1 + -1x3 & 1x2 + -1x4 \end{bmatrix}$

= $\begin{bmatrix} -2 + 6 & -4 + 8 \\ 1 + -3 & 2 + -4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$

Therefore, AB≠ BA

(3) Let A= $\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate AC, BC and (A+B)C and verify that (A+B)C = AC + BC.

Ans: AC = $\begin{bmatrix} 0 & - & 12 & + & 21 \\ -12 & + & 0 & + & 24 \\ 14 & + & 16 & + & 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$

BC = $\begin{bmatrix} 0 & - & 2 & + & 3 \\ 2 & + & 0 & + & 6 \\ 2 & - & 4 & + & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$

A+ B = $\begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

(A +B) C = $\begin{bmatrix} 0 & - & 14 & + & 24 \\ -10 & + & 0 & + & 30 \\ 16 & + & 12 & + & 0 \end{bmatrix}$ = $\begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$

AC + BC = $\begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$

∴ (A +B) C = AC + BC

(4) Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ Show that $A^3 - 23A - 40\,\mathrm{I} = 0$

Ans : $A^3 = A\text{x}\,A\text{x}\,A$

$A^2 = A\text{x}\,A$

$A^3 = A^2\text{ x }A$

$A^2 = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

$A^3 = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$

23A = $\begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$40\,I = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$

$A^3 - 23A - 40\,\mathrm{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

∴ $A^3 - 23A - 40\,\mathrm{I} = 0$

(5) Let A = $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$ What is the value of '$k$' if any make AB = BA

Ans: AB = $\begin{bmatrix} 23 & -10 + 5k \\ -9 & 15 + k \end{bmatrix}$

$$BA = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

AB = BA

-10 + 5$k$ = *15*

*5k = 15 + 10 = 25*

$\therefore k = \frac{25}{5}$ = 5

(6) Two shops have the stock of large, medium and small size of a tooth paste. The number of each size stocked is given by the matrix A, where

$$A = \begin{bmatrix} large & medium & small \\ 150 & 240 & 120 \\ 90 & 300 & 210 \end{bmatrix} \begin{matrix} \\ Shop\ No.1 \\ Shop\ No.2 \end{matrix}$$

are cost matrix 1 of the different size of the tooth paste is given by cost ( ₹)

$$B = \begin{bmatrix} 14 \\ 10 \\ 6 \end{bmatrix} \begin{matrix} Large \\ medium \\ small \end{matrix}$$

Find the investment in the toothpaste by each shop.

Ans : Investment = AB

$$AB = \begin{bmatrix} 150 & 240 & 120 \\ 90 & 300 & 210 \end{bmatrix} X \begin{bmatrix} 14 \\ 10 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2100 & + & 2400 & + & 720 \\ 1260 & + & 3000 & + & 1260 \end{bmatrix} = \begin{bmatrix} 5220 \\ 5520 \end{bmatrix}$$

Investment in toothpaste by

Shop 1 = 5220

Shop 2 = 5520

(7) In a large legislative Assembly electron, a political group hired a public relations firm to promote its candidate in three ways; telephonic, housecalls, and letters. The cost per contract (in paise) is given in matrix A as.

$$A = \begin{matrix} \textit{Cost per Contract} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{matrix} \quad \begin{matrix} Telephone \\ House\ call \\ Letter \end{matrix}$$

The number of contract of each type made in two cities X and Y is given by

$$B = \begin{bmatrix} Telephone & House\ calls & Letter \\ 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$$

Find the total amount spent by the group in the two cities x and y ?

Amount spent = BA

$$BA = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} X \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} 40000 + & 50000 & + & 2{,}50{,}000 \\ 1{,}20{,}000 + & 1{,}00{,}000 & + & 5{,}00{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 3{,}40{,}000 \\ 7{,}20{,}000 \end{bmatrix}$$

Amount spent by

City X = 3,40,000 paise i.e. ₹3400/-

City Y = 7,20,000 paise i.e. ₹7200/-

(8) Three shop keepers A,B and C go to a store to buy stationery. A purchases 12 dozen note book, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen note books, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen note book, 13 dozen pens, and 8 dozen pencils. A note book costs 40 paise, a pen costs ₹1.25 and pencil costs 35 paise. Use matrix multiplication to calculate each individual bill ?

Ans : Bill of purchase = Purchase Quantity x Price

Let A = Purchase Quantity

B = Price

$$\text{Then } A = \begin{bmatrix} Note\ book & Pens & pencil \\ 12 \times 12 & 5 \times 12 & 6 \times 12 \\ 10 \times 12 & 6 \times 12 & 7 \times 12 \\ 11 \times 12 & 13 \times 12 & 8 \times 12 \end{bmatrix} \quad \begin{matrix} Purchase\ A \\ B \\ C \end{matrix}$$

$$\text{Then } A = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix}$$

$$B = \begin{bmatrix} 40 \\ 125 \\ 35 \end{bmatrix} \quad ₹1.25 = 125\ paise$$

$$AB = \begin{bmatrix} 144x40 & + & 60 \text{ x } 120 & + & 72 \text{ x } 35 \\ 120x\ 40 & + & 72 \text{ x } 125 & + & 84 \text{ x } 35 \\ 132 \text{ x } 40 & + & 156 \text{ x } 125 & + & 96 \text{ x } 35 \end{bmatrix}$$

$$AB = \begin{bmatrix} 15780 \\ 16740 \\ 28140 \end{bmatrix}$$

Bill of the Shop keeper A = 15780/-

B = 16740/-

C = 28140/-

**Determinants**

A determinant is a compact form showing a set of numbers arranged in rows and columns, the number of rows and the number of columns being equal. The number in a determinant are known as the elements of the determinant.

Matrics which are not square do not have determinants.

Determinant of Square matrix of order 1

The determinants of 1 x 1 matrix A [a] is denoted by |A| or det. A (i.e. determinant of A) and its value is a.

Determinant of Square matrix of order 2

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2 x 2

Then the determinant A is defined as

|A| = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = ad – bc

Determinant with 3 rows and columns

Let A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a matrix of order 3 x 3.

Then the determinant A is defined as

$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} \; c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ i.e.

***<u>a(ei-uf) - b(di-gf) c (dh-ge)</u>***

***<u>Practical Problems</u>***

1. Evaluate the determinant

$$\begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix}$$

Ans : $\begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix} = 2 \times 9 - 4 \times -3$

$= 18 + 12 \;\underline{\underline{= 30}}$

2. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

Ans : $\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

$1(-4-4) - 2(8-6) - 3(4- -3)$

1 (-8) − 2 (2) – 3 (7)

= -8 – 4 – 21 $\underline{\underline{= -33}}$

Singular and Non singular matrices – A square matrix ‘A’ is said to be singular if its determinant value is zero. If $|A| \neq 0$, then A is called non-singular.

Minor elements of a matrix:

Minor element is the determinant obtained by deleting its rows and the column in which element lies.

Example – (1) Find the Minor of element 6 in the determinant A = $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Ans : Minor of 6 = $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 1 \times 8 - 2 \times 7$

= 8 – 14 $\underline{\underline{= -6}}$

2) If A = $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \\ 4 & 2 & 2 \end{vmatrix}$ Find the minor of 3

Answer : Minor of 3 = $\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$ = 1 x 2 – 0 x 2 = 2 – 0 = 2

**Co-factor of an element**

Co-factor of an element is obtained by multiplying the minor of that element with $(-1)^{(i+j)}$. where i = the row in which the element belongs, s = the column in which the element belongs.

Co-factpr of an element = Minor of an element X $(-1)^{i+j}$

Example 1. Find the Co-factors of all the element of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Ans : Minor element

1 = 3, -2 = 4

4 = -2, 3= 1

Co-factors 1 = $3 \times -1^{1+1} = 3 \times -1^{2}$ = 3

-2 = $4 \times -1^{1+2} = 4 \times -1^{3}$ = -4

4 = $-2 \times -1^{2+1} = -2 \times -1^{3}$ = 2

3 = $1 \times -1^{2+2} = 1 \times -1^{4}$ = 1

2) Find the co-factors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that

$a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

Ans : Minor of an element :

2 = $\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix}$ = (0 x -7) – (4 x 5) = 0 – 20 = -20

-3 = $\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$ = (6 x -7) – (4 x 1) = -42 – 4 = -46

5 = $\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$ = (6 x 5) – (0 x 15) = 30 – 0 = 30

6 = $\begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix}$ = (-3 x -7) – (5 x 5) = 21 – 25 = -4

0 = $\begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix}$ = (2 x -7) – (5 x 1) = -14 – 5 = -19

4 = $\begin{vmatrix} 2 & -3 \\ 1 & -5 \end{vmatrix}$ = (2 x 5) – (-3 x 1) =10 – -3 = 13

1 = $\begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix}$ = (-3 x 4) – (5 x 0) = -12 – 0 = -12

$5 = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$ = (2 x 4) – (5 x 6) = 8 – 30 = -22

$-7 = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix}$ = (2 x 0) – (-3 x 6) = 0 – -18 = 18

Co-factors:

$2 = -20 \times -1^{1+1} = -20 \times -1^{2} = \underline{\underline{-20}}$

$-3 = -46 \times -1^{1+2} = -46 \times -1^{3} = \underline{\underline{46}}$

$5 = 30 \times -1^{1+3} = 30 \times -1^{4} = \underline{\underline{30}}$

$6 = -4 \times -1^{2+1} = -4 \times -1^{3} = \underline{\underline{4}}$

$0 = -19 \times -1^{2+2} = -19 \times -1^{4} = \underline{\underline{-19}}$

$4 = 13 \times -1^{2+3} = 13 \times -1^{5} = \underline{\underline{-13}}$

$1 = -12 \times -1^{3+1} = -12 \times -1^{4} = \underline{\underline{-12}}$

$5 = -22 \times -1^{3+2} = -22 \times -1^{5} = \underline{\underline{22}}$

$-7 = 18 \times -1^{3+3} = 18 \times -1^{6} = \underline{\underline{18}}$

$a_{11} = 2$, $a_{12} = -3$, $a_{13} = 5$

$A_{31} = -12$, $A_{32} = 22$, $A_{33} = 18$

$a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

i.e., $= 2 \times -12 + -3 \times 22 + 5 \times 18$

$= -24 + -66 + 90$

$= \underline{\underline{-90 + 90 = 0}}$

**Adjoint Matrix**

Adjoint of a given matrix is the transpose of the matrix formed by co-factors of the elements. It is denoted by Adj A.

$$\text{Let A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then Adj A} = \text{Transpose} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

**Practical Problems**

1) Find adj A for A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Ans: Minor element:

2 = 4, 3 = 1, 1 = 3, 4 = 2

Co-factors:

2 = 4 x $-1^{1+1}$ = 4, 3 = 1 x $-1^{1+2}$ = -1

1 = 3 x $-1^{2+1}$ = -3, 4 = 2 x $-1^{2+2}$ = 2

$\text{adj A} = \text{Transpose} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

2) Find adj A for A = $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Ans: Minor element:

2 = -1, 1 = 7, 3 = 5

3 = -3, 1 = 3, 2 = 3

1 = -1, 2 = -5, 3 = -1

Co-factor elements

2 = -1 x $-1^{1+1}$ = -1, 1 = 7 x $-1^{1+2}$ = -7

3 = 5 x $-1^{1+3}$ = 5

3 = -3 x $-1^{2+1}$ = 3, 1 = 3 x $-1^{2+2}$ =3

2 = 3 x $-1^{2+3}$ = -3

1 = -1 x $-1^{3+1}$ = -1, 2 = -5 x $-1^{3+2}$ = 5

3 = -1 x $-1^{3+3}$ = -1

$\text{adj A} = \text{Transpose} \begin{bmatrix} -1 & -7 & 5 \\ 3 & 3 & -3 \\ -1 & 5 & -1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$

**Invertible Matrix and Inverse of a Matrix**

Let A be a square matrix of order n, if there exist a square matrix B of order n, such that AB = BA = I

Then A is said to be convertible and B is called on inverse of A and A is called inverse of B

Where I = Identity Matrix

Inverse of A is denoted by $A^{-1}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A \text{ or}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

1) Find the inverse matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Ans: $|A| = (2 \times 3 - 1 \times -1) = 6 - -1 = 7$

Minor element:

2 = 3, -1 = 1, 1 = -1, 3 = 2

Co-factors element

$2 = 3 \times -1^{1+1} = 3$, $-1 = 1 \times -1^{1+2} = -1$

$I = {}^{-}1 \times {}^{-}1^{2+1} = 1$, $3 = 2 \times {}^{-}1^{2+2} = 2$

$$\text{adj } A = \text{Transpose} \begin{bmatrix} 3 & {}^{-}1 \\ 1 & 2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$= \frac{1}{7} \begin{bmatrix} 3 & {}^{-}1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7} & \frac{{}^{-}1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

2. Compute the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ ^-1 & 1 & 1 \end{bmatrix}$

Ans. $|A|$ = 1(3-1) -2(2- ⁻1) + 5 (2-⁻3)

= 1(2) – 2(3) + 5(5)

= 2 – 6 + 25 = 21

Minor element:

1 = 2, 2 = 3, 5 = 5

2 = ⁻3, 3 = 6, 1 = 3

⁻1 = ⁻13, 1 = ⁻9, 1 = ⁻1

Co-factors element

1 = 2 x ⁻1 $^{1+1}$ = 2, 2 = 3 x ⁻1$^{1+2}$ = ⁻3, 5 = 5 x ⁻1 $^{1+3}$ = 5

2 = ⁻3 x ⁻1$^{1+2}$ = 3, 3 = 6 x ⁻1$^{2+2}$ = 6, 1 = 3 x ⁻1$^{2+3}$ = ⁻3

-1= -13 x -1$^{3+1}$ = -13, 1 = -9 x -1$^{3+2}$ = 9, 1 = -1 x -1$^{3+3}$ = -1

$$\text{Adj A} = \text{Transpose} \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adj A})$$

$$= \frac{1}{21} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21} & \frac{3}{21} & \frac{-13}{21} \\ \frac{-3}{21} & \frac{6}{21} & \frac{9}{21} \\ \frac{5}{21} & \frac{-3}{21} & \frac{-1}{21} \end{bmatrix}$$

3) If A = $\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$ then verify that A adjA = |A|1. Also find $A^{-1}$

Ans: |A| = 1(16 – 9) -3(4 – 3) + 3 (3 – 4)

= 1(7) – 3(1) + 3(-1)

= 7 – 3 + -3 = 1

adj A = Transpose $\begin{bmatrix}7 & -1 & -1\\-3 & 1 & 0\\-3 & 0 & 1\end{bmatrix}$

$= \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

A(adj A) $= \begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix} \times \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

A(adj A) = |A|1

$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = 1 \times \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

∴ A (adj A = |A|1

$A^{-1} = \frac{1}{|A|}$ (adj A)

$= \frac{1}{1}\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix} = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

4) If A = $\begin{bmatrix}2 & 3\\1 & -4\end{bmatrix}$ and B = $\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$

Ans: AB = $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ x $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ = $\begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

|AB| = (14 – 25) = -11

$(AB)^{-1} = \frac{1}{|AB|}$ (adj (AB))

adj (AB) = adj A x adj B

adj A:

Minor element:

2 = -4, 3 = 1, 1 = 3, -4 = 2

Co-factors element

2 = -4, 3 = -1, 1 = -3, -4 = 2

adj A = Transpose $\begin{bmatrix} -4 & -1 \\ -3 & 2 \end{bmatrix}$ = $\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

adj B = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{-11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

|A| = -11, |B| = 1

$A^{-1} = \frac{1}{-11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$B^1A^1 = \frac{1}{-11} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$= = \frac{1}{-11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence $(AB)^{-1} = B^1A^1$

**Solving simultaneous equations with the help of Matrices**

Firstly, express the equation in the form of AX = B

Then possibilities

When |A| ≠0

Then $X = A^{-1}B$ i.e., the system has a unique solution.

∴ the system is consistant

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

When $|A| = 0$

Then we calculate (adj A)B

If (adj A)B = 0, then the system will have infinite solution were the system is consistent.

If (adj A)B ≠ 0, then the system will have no solution.

**Problem**

1) Solve the linear equation by using matrix

$5x + 2y = 4$

$7x + 3y = 5$

Ans: $AX = B$

Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$|A| = (15 - 14) = 1$

i.e., $1 \neq 0$

Then $X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

<u>Adj A:</u>

Minor element 5 = 3, 2 = 7, 7 = 2, 3 = 5

Co-factors element 5 = 3, 2 = -7, 7 = -2, 3 = 5

$$\text{adj } A = \text{Transpose} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 12 & -10 \\ -28 & 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

x = 2          y = -3

2) Solve the equation by using matrix

x – y + z = 4

2x + y – 3z = 0

x + y + z = 2

Ans: AX = B

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

|A| = 1(1+3) – (-1)(2+3) +1(2-1)

= 1(4) +1(5) +1(1)

= 4 + 5 + 1 = 10          ie ≠0

Then $X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

Factor elements:

1 = 4,          -1 = -5,          1 = 1

2 = 2,          1 = 0          -3 = -2

1 = 2,          1 = 5,          1 =3

$$\text{Adj A} = \text{Transpose}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}\,(\text{adj A})$$

$$= \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16 + & 0 + & 4 \\ -20 + & 0 + & 10 \\ 4 + & 0 + & 6 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

i.e., x = 2, y = -1, z = 1

3) Solve the following equation by using matrix

5x – 6y + 4 z = 15

7x + 4y – 32 = 19

2x + y + 6 z = 46

Ans: AX = B

Let $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

$|A| = 5(24 - {}^-3) - {}^-6\,(42 - {}^-6) + 4\,(7 - 8)$

$= 5(27) + 6(48) + 4(-1)$

$= 135 + 288 - 4 = 419$

Then $X = A^{-1}B$

$A^{-1} = \frac{1}{|A|}$ (adj A)

Co-factor elements :

5 = 27, -6 = -48, 4 = -1

7 = 40, 4 = 22, -3 = -17

2 = 2, 1 = 43, 6 = 62

$$\text{Adj A} = \text{Transpose}\begin{bmatrix} -27 & -48 & -1 \\ 48 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$X = A^{-1}B$

$X = \frac{1}{|A|}$ (adj A) B

$$= \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix} \times \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$= \frac{1}{419}\begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$X = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$$

∴ x = 3, y = 4, z = 6

# MODULE - II

## THEORY OF EQUATIONS

An equation is a statement of equality between two expressions. For eg:- $x + 2 = 5$. An equation contains one or more unknowns.

**Types of Equations**

**1)Linear Equation**

It is an equation when one variable is unknown. For example $2x + 3 = 7$

### Practical Problems

1) Solve $2x + 3 = 7$

Ans : $2x = 7 - 3$

$2x = 4, x = \frac{4}{2} = 2$

2) Solve $3x + 4x = 35$

Ans : $7x = 35, x = \frac{35}{7} = 5$

3) Solve $4(x - 2) + 5(x - 3) - 25 = x + 8$

Ans : $= 4x - 8 + 5x - 15 - 25 = x + 8$

$= 4x + 5x - x = 8 + 8 + 15 + 25$

$8x = 56$

$x = \frac{56}{8} = 7$

==

4) $7x - 21 - 3x + 13 = 7 + 6x - 19$

Ans : $7x - 3x - 6x =$

$7 - 19 + 21 - 13$

$= {}^{-}2x = {}^{-}4$

$2x = 4$

$x = \frac{4}{2} = 2$

==

5) ${}^{-}23x + 14 - 7x + 16 = 10x - 17 + 3x + 4$

Ans : ${}^{-}23x - 7x - 10x - 3x = 17 + 4 - 14 - 16$

$-23x = -23$

$23x = 23$

$x = \frac{23}{23} = 1$

==

6) Find two numbers whose sum is 30 and difference is 4

Ans : Let one number = $x$

then other number = $30 - x$

Numbers = $(30 - x) - x = 4$

$^-2x = 4 - 30$

$^-2x = -26$

$2x = 26$

$x = \frac{26}{3} = 13$

then numbers are 13, 17

====

7) Two third of a number decreased by 2 equals 4. Find the number

Ans : Let the number = $x$

Then $^2/_3(x) - 2 = 4$

$2x - 6 = 12$

$2x = 12 + 6$

$2x = 18$

$x = 9$

==

8) Solve $\frac{7x+4}{x+2} = \frac{-4}{3}$

Ans : $= 3(7x + 4) = {}^-4\,(x + 2)$

$= (21x + 12) = {}^-\,4x + {}^-8$

$21x + 4x = {}^-8 - 12$

$25x = {}^-20$

$x = \frac{-20}{25} = \frac{-4}{5}$

9) The ages of Hari and Hani are in the ratio of 4 : 5. Eight years from now, the ratio of their ages will be 5:6. Find their present age?

Ans : Let present age = $4x$ and $5\,x$

After 8 years = $\frac{4x + 8}{5x + 8} = \frac{5}{6}$

= 6(4$x$ + 8 ) = 5 (5$x$ + 8)

= 24$x$ + 48 = 25$x$ + 40

= 24$x$ – 25$x$ = 40 – 48

= -1$x$ =-8

= $x$ = 8

Present ages of Hari and Hani are

Hari = 4$x$ = 4 × 8 = 32 years
====

Hani = 5$x$ = 5 × 8 = 40 years
====

**2) Simultaneous equations in two unknowns**

For solving the equations, firstly arrange the equations. For eliminating one unknown variable, multiply the equation 1 or 2 or both of them with certain amount and then deduct or add some equation with another, we get the value of one variable. Then substitute the value in the equation, we get the values of corresponding variable.

***PRACTICAL PROBLEMS***

1) Solve 3x + 4y = 7

4x – 7 = 3

Ans : 3$x$ + 4$y$ = 7 ------ (1)
4$x$ – $y$ = 3 ------- (2)

Multiply the equation 2 by 4, then

3$x$ + 4$y$ = 7 --------- (1)
16$x$ – 4$y$ = 12

Add 19 $x$ = 19

$x = \frac{19}{19} = 1$
==

Substitute to value of $x$

3$x$ + 4$y$ = 7

3 × 1 + 4$y$ = 7

3 + 4$y$ = 7 = 4$y$ = 7 – 3 = 4

$y = \frac{y}{y} = 1$

2) *4x + 2y = 6*
*5x + y = 6*

Ans : $4x + 2y = 6$ ----------- (1)

$5x + y = 6$ ----------- (2)

Multiply the equation 2 by 2, then

$4x + 2y = 6$

$10x + 2y = 12$

$^{-}6x = {}^{-}6$ (Deduct 1 – 2)

$6x = 6$

$x = \frac{6}{6} = 1$

$5x + y = 6$

$5 \times 1 + y = 6$

$5 + y = 6, y = 6 - 5 = 1$

==

Solve $y = 3(x + 1)$

$4x = 4 + 1$

*Ans :* $y = 3x + 1$

$4x = 4 + 1$

Arrange the equation

$-3x + y = 3$ ----- (1)

$4x - y = 1$ ---------- (2)

$1x = 4$ *Add*

$x = 4$

Substituting the value of $x$

$4x - y = 1$

$16 - y = 1$

$Y = 16 - 1 = 15$

$X = 4, \ y = 15$

=========

4) Solve $8x + 7y = 10$

$11x = 10(1-y)$

Ans : $8x + 7y = 10$ ------ (1)

$11x = 10 - 10y$

$11x + 10y = 10$ --------- (2)

Multiply equation (1) by 11 and (2) by 8

*88x + 77y = 110*
*88x + 80 y =80*

---

(1-2) *-3y = 30*

$y = \frac{30}{-3} = {}^{-}10$

Substituting the value of *y*

*8x + 7y = 10*

*8x + 7 × -10 = 10*

*8x + -70 = 10*

*8x = 10 + 70*

$8x = 80,\ x = \frac{80}{8} = 10$

*x = 10, y = -10*

===========

5) Solve $\frac{x-y}{2} = \frac{y-1}{3}$ and $\frac{3x-4y}{5} \times {}^{-}10$

$\frac{x-y}{2} = \frac{y-1}{3}$

*= 3(x-y) = 2 (y-1)*

*= 3x – 3y = 2y – 2*

*3x – 3y – 2y = -2*

*3x – 5y = -2 ---------- (1)*

$\frac{3x-4y}{5} = x - 10$

*3x – 4 y = 5 (x-10)*

*3x – 4y = 5x – 50*

*3x -5x – 4y = -50*

*= 2x + 4y = 50*

*= x + 2y = 25 -------- (2)*

*Multiply equation (2) by 3*

*3x – 5y = ⁻2*

*3x + 6y = 75*

*(1-2) -11y = -77*

$y = \frac{-77}{-11} = 7$

*Substituting the value*

$x + 2y = 25$

$x + 2y = 25$

$x = 11$
====

$x = 11, y = 7$
=======

6) A man sells 7 horses and 8 cows at Rs. 2940/- and 5 horses and 6 cows at Rs. 2150/-. What is selling price of each ?

Ans : Let the selling price of horse = $x$
Cow = $y$

$7x + 8y = 2940$ ------- (1)

$5x + 6y = 2150$ --------(2)

Multiply equation (1) by 5 and 2 by 7

Then $3x + 40y = 14700$

$35x + 42y = 15050$

(1-2) $-2y = -350$

$$y = \frac{-350}{-2} = 175$$

Substituting the value of $y$

$7x + 8y = 2940$

$7x + 8 \times 175 = 2940$

$7x = 2940 - 1400$

$7x = 1540$

$$x = \frac{1540}{7} = 220$$

Selling price of horse = 220
===

Selling price of cow = 175
===

**3) Simultaneous Equations in three unknowns**

Firstly, eliminate one of the unknown from first two equations. Then eliminate the same unknown from second and third equations. Then we get two equations. Solve such equations, we get the values of *x, y* and *z*.

1) Solve $4x + 2y - 32 = 2$

$3x + 4y - 2z = 10$

$2x - 5y = 5$

Ans: First consider first two equation and eliminate one unknown

$4x + 2y - 3z = 2$

$3x + 4y - 2z = 10$

For eliminating 2 multiply equation in 1 by 2 and 2 by 3, then

$8x + 4y - 62 = 4$

$9x + 12y = 30$

(2-1) $x + 8y = 26$ ________ (1)

Consider equation 2 and 3

$3x + 4y - 2z = 10$

$2x - 5y + 4z = 5$

On multiply xy equals 2 by 2

$6x - 8y - 42 = 20$

$2x - 5y + 42 = 5$

add $8x + 3y = 25$ ________ (2)

Solve the new equation 1 and 2

$x + 8y = 26$ ________(1)

$8x + 3y = 25$ ________ (2)

Multiply equation 1 by 8, then

$8x + 64y = 208$

$8x + 3y = 25$

(1-2) $61y = 183$

$Y = \frac{183}{61} = 3$

Substitute value of Y

$x + 8y = 26$

$x + 8 \times 3 = 26$

$x + 24 = 26$

$x = 26 - 24 = 2$

Substitute the value of x, y,

4x + 2y – 3z = 2

4 x 2 + 2 x -3z = 2

8 + 6 – 3z = 2

14 – 3z = 2

3z = 14 – z

3z = 12

z = 12/ 3 = 4

x = 2, y=3, z=4

=============

**4) Quadratic equations**

The equation of the form $ax^2$ + bx+ c = 0 in which a, b, c are constant is called a quadratic equation in x. Here x is the unknown.

Solution of quadratic equations

There are three methods to solve a quadratic equation.

(1) Method by formula
(2) Method of factorization
(3) Method of completing the squre

<u>Quadratic formula method</u>

One general quadratic equation is $ax^2$ + bx + c = 0

Then $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

1) Solve the equation $x^2 - \text{x} - 12 = 0$

Ans: a = 1, b= -1, c=-12

$x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

$\frac{\sqrt{-(-1)^2 \pm 4 \times 1 \times (-12)}}{2 \text{ x } 1}$

$1 \pm \frac{\sqrt{49}}{2}$

$1 \pm \frac{7}{2} = \frac{8}{12}, or \ \frac{-6}{2}$

4 or -3

=======

2) Solve the equation 2x + $\frac{5}{x}$ = 7

Ans: Multiply the equation by x

Then

$2x^2 + 5 = 7x$

$2x^2 - 7x + 5 = 0$

a = 2, b= -7, c=5

$x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

$-(-1) \pm \frac{\sqrt{1-4 \ \ 1 \text{ x} 1 \text{ x } -7}}{2 \text{ x } 1}$

$1 \pm \frac{\sqrt{1- - \ 28}}{2}$

$1 + \frac{\sqrt{29}}{2}$ = $1 \pm \frac{\sqrt{29}}{2}$

3) Solve the equation ( x + 1) (x +2) − 3 = 0

Ans: $x^2$ + 2x + x + 2 – 3 = 0

$x^2$ +3x + 2 − 3 = 0

$x^2 + 3x - 1 = 0$

a = 1, b = 3, c= −1

$x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

$-3 \pm \frac{\sqrt{3^2-4 \text{ x } 1 \text{ x} 1}}{2 \text{ x } 1}$

$-3 \pm \frac{\sqrt{9 \ - -4}}{2}$ = $-3 \pm \frac{\sqrt{13}}{2}$

========

4) Solve $x^4$ - 10 $x^2$ + 9 = 0

Ans: Let $x^2$ = y

Then equation =

$y^2 - 10y + 9 - 3 = 0$

$y = -b \pm \frac{\sqrt{b^2-4ac}}{1}$

a = 1, b = -10, c= 29

$= (-10 \pm \frac{\sqrt{-10^2-4 \text{ x } 1 \text{x } 9}}{2}$

$10 \pm \frac{\sqrt{100-36}}{2}$

$10 \pm \frac{8}{2}$ = 9, 1

Y = 9, 1

$x^2$ =y , then x = $\sqrt{y}$

Y = 1, x= $\sqrt{1}$ = $\pm$ 1

Y = 9, x =$\sqrt{9}$ = $\pm$ 3

X = -1, 1, 3, -3

===========

5) 2x-7$\sqrt{x}$ + 5 = 0

Answer = Let$\sqrt{x}$ = y, then equation

$2y^2$ – 74 + 5 = 0

y = -b $\pm \frac{\sqrt{b^2-4ac}}{2a}$

-7$\pm \frac{\sqrt{-7^2-4x2x5}}{2 \text{ x } 2}$

7$\pm \frac{\sqrt{49-40}}{4}$

7$\pm \frac{3}{4} = \frac{10}{4}$ or

y = 4, or $\frac{-1}{2}$

y = 1, x = $1^2$ = 1

y = $\frac{10}{4}$ = x = $\frac{10^2}{4^2} = \frac{100}{16} = \frac{25}{4}$

x= 1, $\frac{25}{4}$

===========

6) Solve $x^{10} - 33x^5 + 32 = 0$

Ans: Let y = $x^5$, Then equation

= $y^2 - 33y + 32 = 0$

Use quadratic formula

Y = 32, 1

Y = 32 then $x^2$ = 32

= $2^5$=32

∴ x = 2

y = 1 then $x^5 = 1$

= $1^5$ = 1, x = 1

X = 2, 1

======

7) Solve x + y = 10

xy = 24

Ans: change to equation in the form of quadratic

x + y = 10

x= 10-y

Substitute the value in second equation

xy = 24

(10-y) y = 24

= 10y $-y^2$ = 24

$y^2 - 10 + 24 = 0$

Use quadratic formula

$y= -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

$= (-10 \pm \frac{\sqrt{-10^2-4 \text{ x } 1 \text{x } 24}}{2 \text{ x } 1}$

$10 \pm \frac{\sqrt{100-96}}{2}$

$10 \pm \frac{2}{2}$ = 6, 4

when y = 6, x = 4

y= 4, x = 6

8) Simultaneous equations of two unknowns when one of them is quadratic and the other is linear

1) x + y = 7

$x^2 + y^2 = 25$

Answer

x + y = 7

y = 7 – x

Substistue the value of y y in the second equation, then

$x^2 + (7-x)^2 = 25$

We know $(a-b)^2 = a^2 - 2ab + b^2$

$x^2 + 7^2 - 2 \text{ x } 7 \text{ x } x + x^2 = 25$

$x^2 + 49 - 14x + x^2 = 25$

$x^2 + x^2 - 14x + 49 - 25$

$2x^2 - 14x + 24 = 0$

Use quadratic formula

$y = -b \pm \frac{\sqrt{b^2-4ac}}{2a} =$

$=$

$= (-14 \pm \frac{\sqrt{-14^2-4 \text{ x } 2 \text{x } 24}}{2 \text{ x } 2}$

$14 \pm \frac{\sqrt{4}}{4} = 14 \pm \frac{2}{4} = 4, 3$

When y= 4, x = 3

Y=3, x=4

=========

2) Solve x + y = 5

$2x^2 - y^2 - 10x - 2xy - 28 = 0$

Ans: y = 5- x

Substitute the value of y is equation (2)

$2x^2 - (5-x)^2 - 10x - 2x(5-x)$

$+ 28 = 0$

$= 3x^2 \ 10x + 3 = 0$

Use quadratic formula

$X = 3 \text{ or } \frac{1}{3}$

When x = 3, y=2

When $x = \frac{1}{3}, y = \frac{14}{3}$

# MODULE – III

## PROGRESSIONS

### Arithmetic Progression

A series is said to be in Arithmetic Progression, it its terms continuously increase or decrease by a constant number . It is a series, in which each term is obtained by adding or deducting a constant number to the preceding term. The constant number is called common difference of the progression and is denoted by 'd'. It is the difference between the two term of the series i.e., the difference between second term and first term or third term and second term and so on.

The first term of an A.P. is usually denoted by 'a'. One general form of an A.P is *a*, *a+ d, a+2d, a+ 3d*, ………

For example

(i) The sequence 1, 3, 5, 7, . . . . . . is an A.P whose first term is 1 and d = 2

(ii) The sequence -5, -2, 1, 4, 7, . . . . , whose 'a' = -5, d = 3

### General term of an AP or $n^{th}$ term

Ler '*a'* be the first term and '*d'* be the common difference of an A.P, then $a_n$ denotes the $n^{th}$ term of the A.P.

$$a_n = a + (n-1)d$$

n = number of term in a series.

### Practical Problems

1) Find the 12th term of an A.P 6, 2, -2

Ans: $a_n = a + (n-1)d$

a = 6, n = 12, d = -4

= 6 + (12-1) – 4

= 6 + (11) – 4

= 6 + $^-$44 = -38

12th term is -38

2) Find the 8th term of the series 6, 5½, 5, 4½, . . . . .

Ans: a = 6, d = -½,n = 8

$a_n = a + (n-1)d$

$= 6 + (8-1)\text{-}½$

$= 6 + (7)\text{ -}½$

$= 6 + -3.5 = 2.5$

3) Which term of the A.P 21, 18, 15, . . . . . . -81 ?

Ans: a = 21, d = -3, $a_n = -81$ n = ?

$a_n = a + (n-1)d$

-81 = 21 + (n-1)-3

-81 = 21 + -3n +3

-81 = 24 – 3n

-81 – 24 = -3n

3n = 105

n = 105/3 = 35

Therefore the 35th term of the given A.P = -81

4) Which term of the A.P 21,18,15, . . . . . 0 ?

Ans: a = 21, d = -3, $a_n = 0$, n = ?

$a_n = a + (n-1)d$

0 = 21 + (n – 1)-3

0 = 21 + -3n + 3

0 = 24 – 3n

3n = 24, n = 8

Therefore, the 8th term = 0

5) If the 9th term of an A.P is 99 and 99th term is 9. Fine 108th term?

Ans: $a_n = a + (n-1)d$

n = 9, $a_n = 99$

= a + (9-1)d = 99

= a + 8d = 99 ----------------(1)

n = 99, $a_n = 9$

= a + (99 – 1)d =9

= a + 98d = 9 ---------------(2)

Solve the equations

$a + 8d = 99$ ----------------(1)
$a + 98d = 9$ ----------------(2)

Then (1) – (2) $-90d = 90$

$d = 90/-90 = -1$

Substitute the value of 'd'

$a + 8d = 99$

$a + 8 \times -1 = 99$

$a + -8 = 99$

$a = 99 + 8 = 107$

$108^{th}$ term = $a + (n-1)d$

$= 107 + (108 - 1)-1$

$= 107 + (107)-1$

$= 107 - 107 = 0$

$108^{th}$ term = 0

6) Determine the A.P whose $3^{rd}$ term is 5 and the $6^{th}$ term is 8

Ans: $a + 2d = 5$ --------------------- (1)
$a + 5d = 8$ ---------------------(2)

Then (1) – (2) = $-3d = -3$

$$d = \frac{3}{3} = 1$$

A.P = 3, 4, 5, 6, 7, 8 ………………

7) Find many two digit numbers are divisible by 3 ?

Ans: Numbers = 12, 15, 18, - - - - - - 99

$a = 12$, $d = 3$, $a_n = 99$

$a_n = a + (n-1)d$

$99 = 12 + (n-1)3$

$99 = 12 + 3n - 3$

$99 = 12 - 3 + 3n$

$99 = 9 + 3n$

$3n = 99 - 9$, $3n = 90$

$$n = \frac{90}{3} = 30$$

∴ Two digit numbers are divisible by 3 = 30 number

8) Determine the $25^{th}$ term of the A.P, whose $9^{th}$ term is -6 and the common difference is 5/4.

Ans: d = 5/4, $a_9$ = -6

$a_9 = a + (n-1)d$

$-6 = a + 8 \times \frac{5}{4}$

$-6 = a + 10$

$a = -10 - 6 = -16$

$a_{25} = a + (n - 1)d$

$= -16 + (25 - 1) \frac{5}{4}$

$= -16 + 24 \times \frac{5}{4}$

$= -16 + 30 = 14$

$25^{th}$ term = 14

**Sum of n terms of an A.P**

Let $S_n$ denotes the sum of '*n*' terms of an A.P, whose first term is '*a*' and common difference is '*d*'.

$$S_n = n/2\,[2a + (n-1)d]$$

$2a = a + a$ or $2 \times a$

**Practical Problems**

(1) Find the sum of the first 20 terms of 1 + 4 + 7 + 10 . . . . . . .

Ans: $S_n = n/2\,[2a + (n-1)d]$

n = 20, $a$ = 1, d = 3

$S_n = \frac{20}{2} (2 \times 1 + (20 - 1)3$

$= 10\ (2 + 19 \times 3)$

$= 10(2 + 57)$, $10 \times 59 = 590$

Sum of the first 20 terms = 590

2) Find the sum of the series 5, 3, 1, -1, . . . . . . -23

Ans: $a$= 5, d = -2, n = ?, $a_n$ =-23

$S_n = {}^{n}/_{2}\,[2a + (n-1)d]$

We know, $a_n = a + (n - 1)d$

-23 = 5 + (n – 1)-2

-23 = 5 + -2n + 2

-23 = 5 + 2 – 2n

-23 = 7 – 2n

2n = -23 – 7

2n = 30, n = $\frac{30}{2}$ = 15

$S_n = \frac{15}{2}$ (2 x 5 + (15 – 1)-2)

= $\frac{15}{2}$ (10 + 14 x-2)

= $\frac{15}{2}$ (10 + -28)

= $\frac{15}{2}$ x -18 = 15 x -9 = -135

Sum of the series = -135

3) How many terms of the sequence 54, 51, 48, ………… be taken so that their sum is 513. Explain the double answer.

Ans: $S_n$ = 513, $a$ = 54, d = -3

$S_n = {}^{n}/_{2}\,[2a + (n-1)d]$

513 = $\frac{n}{2}$ (2 x 54 + (n – 1)-3)

513 = $\frac{n}{2}$ (108 – 3n + 3)

513 = $\frac{n}{2}$ (111 – 3n)

= 1026 = n(111 – 3n)

= 1026 = 111n – $3n^2$

= $3n^2$ – 111n = -1026

= $3n^2$ – 111n + 1026 = 0

$= n^2 - 37n + 342 = 0$

Solve by using quadratic formula

i.e., $n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$, b = -37, c = 342

$n = \frac{37 \pm \sqrt{37^2 - 4x1x342}}{2x1}$

$= \frac{37 \pm \sqrt{1369 - 1368}}{2}$

$= \frac{37 \pm \sqrt{1}}{2} = \frac{37 \pm 1}{2}$

$= \frac{37+1}{2}$ or $\frac{37-1}{2}$

= 19 or 18

N = 18 or 19

4) Find the sum of all natural numbers between 500 and 1000 which are divisible by 13.

Ans: Number between 500 and 1000 which are divisible by 13

507, 520, 533, . . . . 988

$a = 507$, d = 13, $a_n = 988$

$a_n = a + (n-1)d$

988 = 507 + (n-1)13

988 = 507 + 13n – 13

988 = 507 – 13 + 13n

988 = 494 + 13n

13n = 988 – 494 = 494

13n = 494

$n = \frac{494}{132} = 38$

$S_n = {}^{n}/_{2}\,[2a + (n-1)d]$

= 19(1014 +37 x 13)

= 19(1014 + 481)

= 19 1495 = 28405

5) Find the sum of all natural numbers from 1 to 200 excluding those divisible by 5

Ans: Natural number from 1 to 200 = 1, 2, 3, 4, ……. 200

Divisible by 5 = 5, 10, 15, 20 . . . . . 200

∴ Natural numbers from 1 to 200, excluding divisible by 5 =

(1, 2, 3, 4 . . . . 200) – (5, 10, 15 . . . . . 200)

Sum of (1, 2, 3, 4, . . . . . 200) =

$S_n = {}^{n}/_{2}\,[2a + (n-1)d]$

$= \frac{200}{2}\left[2\text{x}1 + (200-1)1\right]$

= 100 (2+199)

=100 x 201 = 20,100

Sum of (5, 10, 15,20, . . . . . . 200)

$= \frac{40}{2}\,(2\text{ x}5 + (40\text{-}1)5)$

=20 (10 + 39 x 5)

=20(10 + 195)

= 20 x 205 = 4100

Sum by natural numbers from 1 to 200 excluding divisible by 5 = 20100 – 4100

= 16000

6) The sum of the first 3 terms of an A.P is 30 and the sum of first 7 terms is 140. Find the sum of the first 10 terms.

Ans: $S_3 = 30$, $s_7 = 30$,

$S_n = {}^{n}/_{2}\,[2a + (n-1)d]$

$= \frac{3}{2}\left[2a + (3-1)d\right] = 30$

$= 2a + 2d = 30 \text{ x} \frac{2}{3}$

= 2a + 2d = 20

= a + d = 10 --------------(1)

$= \frac{7}{2}\left[2a + 6d\right] = 140$

$= 2a+6d = 140 \times \frac{2}{7}$, $= 2a + 6d = 40$

a + 3d = 20 ---------------(2)

Solving the equation (1) and (2) d = 5

Then a = 5

$S_{10} = \frac{10}{2}\left[2 \times 5 + 9 \times 5\right] = \underline{\underline{275}}$

7) Find three numbers in A. P whose sum is 9 and the product is -165.

Ans: Let the numbers be a-d, a, a+d

(a-d) + a + (a +d) = 9

3a = 9, a = 3

$(a$-d$)$ x $a$ x $(a$+d) = -165

= (3 – d) x 3 x (3 +d) = -165

$= 9 - d^2 = \frac{-165}{3}$

$= 9 - d^2 = -55$

$= -d^2 = -55 - 9 = -64$

$= d^2 = 64$, d = 8

$a = 3$, d = 8

Numbers = (a – d), a, (a+d)

= $\underline{\underline{-5, 3, 11}}$

8) Find four numbers of A.P whose sum is 20 and the sum of whose square is 120

Ans: Let numbers be (a-3d), (a-d) (a+d) (a+3d)

Given (a-3d)+(a-d)+(a+d)+(a+3d) = 20

4a = 20, $a = \frac{20}{4} = 5$

$(a-3d)^2 \times (a-d)^2 \times (a+d)^2 \times (a+3d)^2 = 120$

$= (5-3d)^2 \times (5-d)^2 \times (5+d)^2 \times (5+3d)^2 = 120$

We know $(a-b)^2 = a^2 - 2ab + b^2$

$= 25 - 30d + 9d^2 + 25 - 10d + d^2 + 25 + 10d + d^2 + 25 + 30d + 9d^2 = 120$

$= 100 + 20d^2 = 120$

$20d^2 = 120 = 100$

$20d^2 = 20,\quad d^2 = 20/20 = 1,\quad d = 1$

$a = 5,\quad d = 1$

Numbers are = (a – 3d), (a-d), (a + d), (a + 3d)

= (5-3). (5-1), (5+1), (5+3)

` = 2, 4, 6, 8

9) A manufacturing of radio sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the production uniformly increases by a fixed number every year. Find

(1) One production in the first year

(2) The production in the 10th year.

(3) The total production in 7 year.

Ans: Since the production increases uniformly by a fixed number in every year, it form an A.P.

Let $a_3 = 600,\quad a_7 = 700$

$a_n = a + (n - 1)d$

$600 = a + (3-1)d$

$600 = a + 2d$ …………………… (1)

$700 = a + 6d$ …………………… (2)

$a + 2d = 600$ ……………………….(1)
$a + 6d = 700$ ……………………….(2)

$-4d = -100$

$$d = \frac{100}{4} = 25$$

(1) Production in the first year

$a + 2d = 600$
$a + 50 = 600$
$a = 550$

(2) Production in the 10th year

i.e., $a_n = a + (n-1)d$
$= 550 + (10 - 1)\ 25$
$= 550 + 9 \times 25$
$= 550 + 225 = 775$

(3) Total production in 7th year

$S_n = {n}/{2}\,[2a + (n-1)d]$

$= \frac{7}{2}\left[2 \times 550 + (7-1)25\right]$

$= \frac{7}{2}(1100 + 6 \times 25)$

$= \frac{7}{2}(1100 + 150)$

$= \frac{7}{2}(1250)$

$= 7 \times 625 = \underline{\underline{4375 \text{ units}}}$

10) The rate of monthly salary of a person is increased annually in A.P. It is known that he was drawing as 400 a month during the 11th year of his service and as 760 during the 29th year. Find

(1) Starting salary

(2) Annual increment

(3) Salary after 36 years.

Ans: $a_{11} = 400$

$a_{29} = 760$

$a + 10d = 400$

$a + 28d = 760$

$-18d = -360$

$d = \frac{360}{18} = 20$

$a + 10d = 400$

$a + 10 \times 20 = 400$

$a + 200 = 400$

$a = 400 - 200 = 200$

$a_{36} = 200 + 35d$

$200 + 35 \times 20$

$200 + 700 = \underline{\underline{900}}$

1) Starting salary = 200
2) Annual Increment = 20
3) Salary after 36 years = 900

**Arithmetic Mean (A.M)**

Given two numbers a and b, we can insert a number A between them, so that a, A, b is an A.P. Such a number A is called the Arithmetic Mean of the number a and b.

We can insert as many numbers as we like between them. Let A, $A_2$, $A_3$ . . . . . $A_n$ be 'n' numbers between a and b,

Then

$A_1 = a + d$

$A_2 = a + 2d$

$A_3 = a + 3d$

$A_n = a + nd$

Example

1) Find A.M between 2 and b

Ans: A.M between 2 and 6 = $\frac{2+6}{2}$ = 4

Then A.P. = 2, 4, 6

2) Insert 4 Arithmetic means between 5 and 20

a = 5, n = 6, $a_n$ =20, d = ?

$a_n = a + (n - 1)d$

20 = 5 + (6-1)d

20 = 5 + 5d

20 = 5 + 5d

5d = 20 – 5 = 15

d = 15/5 = 3

$A_1 = a + d$ i.e., 5 +3 = 8

$A_2 = a + 2d$ i.e., 5 +6 = 11

$A_3 = a + 3d$ i.e., 5 +9 = 14

$A_4 = a + 4d$ i.e., 5 +12 = 17

Arithmetic means are 8, 11, 14, 17

A.P. = 5, 8, 11, 14, 17, 20

3) Insert six numbers between 3 and 24 such that the resulting sequence is an A.P.

Ans: a = 3, n = 8, $a_n = 24$, d = ?

$a_n = a + (n - 1)d$

24 = 3 + 7d

7d = 21, d = 3

$A_1 = 3 + 3 = 6$

$A_2 = 3 + 6 = 9$

$A_3 = 3 + 9 = 12$

$A_4 = 3 + 12 = 15$

$A_5 = 3 + 15 = 18$

$A_6 = 3 + 18 = 21$

A.M. = 6, 9, 12, 15, 18, 21

A.P. = 3, 6, 9, 12, 15, 18, 21, 24

**Geometric Progression**

A series is said to be in G.P if every term of it is obtained by multiplying the previous term by a constant number. This constant number is called common ratio, denoted by 'r'. r = $\frac{second\ term}{first\ term}$ or third term by second term etc.

The first term of a G.P is usually denoted by a. The general form of a G.P is usually denoted by a. The general form of a G.P is a, ar, $ar^2$, $ar^3$ ….. If the number of terms of a G.P is finite, it is called a finite G.P, otherwise it is called an infinite G.P. For example.

(i) 1, ½, ¼, $^1/_8$ ……… is a G.P, whose first term is 1 and r = ½

(ii) 3, -6, 12, -24 ………… is a G.P whose a = 3 , r = -2

**General term of a G.P or $n^{th}$ term of a G.P**

Let 'a' be the first term and 'r' be the common ratio of a G.P, then

$$a_n = ar^{n-1}$$

1) Find $10^{th}$ term of series 9, 6, 4………

Ans: a = 9, $r = \frac{6}{9} = \frac{2}{3}$, n = 10

$a_n = ar^{n-1} = 9 \times (^2/_3)^{10-1}$

$= 9 \times (^2/_3)^9 = 9(^2/_3)^9$

2) Find the 12th term of 2, 6, 18, 54 .........

$a = 2, \quad r = 6/2 = 3, \quad n = 12$

$a_n = ar^{n-1} = 2 \times 3^{12-1}$

$= 2 \times 3^{11} = 2 \times 177147 = \underline{\underline{3,54,294}}$

3) Which term of the G.P 2, 8, 32 .......... Up to n terms is 131072 ?

$a = 2, \quad r = 4, \quad a_n = 1,31,072$

$a_n = ar^{n-1}$

$1,31,072 = 2 \times 4^{n-1}$

$4^{n-1} = \frac{1,31,072}{2} = 65536$

$4^{n-1} = 65536$

i.e., $4^8 = 65536$

i.e. $n-1 = 8$

$\therefore n = 8 + 1 = 9$

Hence 1,31,072 is the 9th term of the G.P.

4) In a G.P the third term is 24 and 6th term is 192. Find the 10th term .

Ans: $a_3 = 24, \quad a_6 = 192$

$a_n = ar^{n-1}$

$a_3 = ar^2 = 24$

$a_6 = ar^5 = 192$

i.e., $ar^2 = 24$ --------- (1)

$ar^2 = 192$ -------(2)

Divide (2) by (1),

$\frac{ar^5}{ar^2} = \frac{192}{24}$

$r^3 = 8$ i.e., $2^3$

$r = 2$

Substituting r = 2 in (1)

$ar^2 = 24, \quad a \times 2^2 = 24$

$a \times 4 = 24, \quad a = 24/4 = 6$

$a_{10} = ar^{n-1} = 6(2)^9 = \underline{\underline{3072}}$

**Sum of 'n' terms of a G.P**

Let 'a' be the first term and 'r' be the common ratio and $S_n$ the sum of the 'n' terms of G.P.

Then $S_n = \frac{a(1-r^n)}{(1-r)}$ or $\frac{a(r^n-1)}{(r-1)}$

When r is less than 1, we can apply first formula.

1) Find the sum of the series.

1024 + 512 + 256 .........to 15 terms

Asn: a = 1024, n = 15, r = ½

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$= \frac{1024(1-½^{15})}{(1-½)}$$

$$= \frac{1024\times (1/2)^{15}}{(1-½)}$$

$$= 1024 \quad \frac{2}{1} \times \left(\frac{1}{2}\right)^{15}$$

$$= 2048 \times \left(\frac{1}{2}\right)^{15}$$

2) Find the sum of 1 + 3+ 9 + 27 ....... to 10 terms.

a = 1, r = 3, n = 10

$$S_n = \frac{a(r^n-1)}{(r-1)}$$

$$= \frac{1(3^{10}-1)}{(3-1)} = \frac{59049-1}{2} = 29524$$

3) How many terms of the G.P 3, 3/2, 3/4 , …………….. are needed to give the sum $\frac{3069}{512}$

Ans: a = 3, r = ½, $S_n = \frac{3069}{512}$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\frac{3069}{512} = \frac{3(1-(½)^n)}{(1-½)}$$

$= \frac{3069}{512} = \frac{3(1-(½)^n)}{½}$

$= \frac{3069}{512} = 3 \text{ x } \frac{2}{1} (1- \left(\frac{1}{2}\right)^n$

$= \frac{3069}{512} = 6 \ (1- \left(\frac{1}{2}\right)^n$

$= \frac{3069}{512x6} = (1- \left(\frac{1}{2}\right)^n = \frac{3069}{3072} = 1- \frac{1}{2^n}$

$= \frac{1}{2^n} = 1- \frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$

$2^n = 1024$

$2^{10} = 1024$, n = 10

4) Find three numbers in G.P whose sum is 14 and product is 64

Ans: Let the numbers = a/r, a, ar

$\therefore \frac{a}{r} + a + ar = 14$

$\frac{a}{r}$ x a x ar = 64

$\therefore a^3 = 64$

$4^3 = 64$

$a = 4$

Substituting value of $a$

$\frac{a}{4} + a + ar = 14$

$\frac{4}{r} + 4 + 4r = 14$

Multiply by 'r'

Then = $4 + 4r + 4r^2 = 14r$

$4r^2 - 10r + 4 = 0$

Use quadratic formula, for getting the value of 'r'

r = 2 cor ½

numbers = $\frac{a}{r}, a, ar$

r = 2 = $\frac{4}{2}$, 4, 4 x 2, r = ½ = 8, 4, 2

= 2, 4, 8

Both are the same = 2, 4, 8

5) A Person has 2 parents, 4 grand parents, 8 great grant parents and so on. Find the number of his ancestors during the ten generations preceding his won.

$a = 2, \quad r = 2, \quad n = 10$

$$s_n = \frac{a\,(r^n - 1)}{r - 1}$$

$$s_{10} = \frac{2\,(2^{10} - 1)}{2 - 1}$$

$$= \frac{2\,(2^{10} - 1)}{1}$$

$$= 2\,(2^{10} - 1) = 2046$$

Number of ancestors preceding the person is 2046.

## Geometric Mean

One geometric mean of two positive numbers $a$ and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We can insert as many numbers as we like between $a$ and b to make the sequence in a G.P. Let $G_1$, $G_2$, $G_3$, ………… $G_n$ be 'n' number between a and b, then

$G_1 = ar$, $G_2 = ar^2$, $G_3 = ar^3$, $G_n = ar^n$

1) Insert three G.M. between 1 and 256

Ans. $a = 1$, $a_n = 256$, $n = 5$, $r = ?$

$$a_n = ar^{n-1}$$

$$256 = 1\,r^{n-1}$$

$$256 = r^{n-1}$$

$$256 = r^{5-1}$$

$$256 = r^4$$

$256 = 4^4$, $r = 4$

G.M. are $ar$, $ar^2$, $ar^3$

$1 \times 4$, $1 \times 4^2$, $1 \times 4^3$ = 4, 16, 64

G.P = 1, 4, 16, 64, 256

2) Find the G.M between 4 is 16

Ans: G.M = $\sqrt{4 \times 16}$ = $\sqrt{64}$ = 8

3) Insert 5 geometric means between 2 and 1458

Ans: a = 2, n = 7, $a_n$ = 1458

$a_n = ar^{n-1}$

$1458 = 2\, r^{7-1}$

$1458 = 2r^6$

$2r^6 = 1458$

$r^6 = \frac{1458}{2}$ , $r^6 = 729$

$r^6 = 3^6$

∴ r = 3

G.M. = ar, $ar^2$, $ar^3$, $ar^4$, $ar^5$

= 2 x 3, 2 x $3^2$, 2 x $3^3$, 2 x $3^4$, 2 x $3^5$

= 6, 18, 54, 162, 486, 486

G.P. = 2, 6, 18, 54, 162, 486, 1458

4) If the A.M. between two positive numbers is 34 and their G.M. is 16. Find the numbers?

Ans: Let the numbers a and b

A.M. = $\frac{a+b}{2}$ = 34

G.M = $\sqrt{ab}$ = 16

∴ $a$ +b = 68

$a$ x b = 256

b = 68 – $a$

$a$b = 256

$a(68-a) = 256$

$a^2 - 68a = 256$

$a^2 - 68a - 256 = 0$

Using quadratic formula

$a$ = 4 or 64

When $a$ = 4, b = 64

When $a$ = 64, b = 4

Required numbers are 64 and 4

5) Find the three numbers in G.P whose sum is 26 and product is 216.

Ans: Let the number is G.P be

a/r, a, ar

a/r, a/ ar = 216

i.e. $a^3 = 216$, $6^3 = 216$

∴ a = 6

a/r + a + ar = 6/r + 6 + 6r = 26

= 6/r + 6r = 26 – 6

= 6/r + 6r = 20

Multiply by r

$= 6 + 6^2 = 20r$

$= 6^2 - 20r + 6$

$= 6^2 - 20\,r + 6 = 0$

Solving by using quadratic formula

Then r = 1/3 or 3

Required numbers a/r, a, ar

r = 3

6/3, 6, 6 x 3 = 2, 6, 18

# MATHEMATICS OF FINANCE

## Simple interest

It is the interest calculated on principal amount at the fixed rate .

Simple Interest = $\frac{Pnr}{100}$

Where P = Principal amount, n = number of year,

r = rate of interest per annum

Amount at the end of $\mathbf{n^{th}}$ year = $P + \frac{Pnr}{100}$ or

$P(1 + \frac{nr}{100})$

or principal amount + interest

1) What is the simple interest for Rs. 10, 000 at the rate of 15% per annum for 2 years?

Ans: P = 10,000, n = 2 years, r = 15

Interest = $\frac{Pnr}{100} = \frac{10,000 \times 2 \times 15}{100}$

= Rs. 3, 000

2) Find the total interest and amount of the end of $5^{th}$ year for as 10,000 at 10% per annum, simple interest.

Ans: P = 10,000, n = 5 years, r = 10%

Interest = $\frac{Pnr}{100} = \frac{10,000 \times 5 \times 10}{100}$

= Rs. 5, 000

Amount at the end

$5^{th}$ year = $P(1 + \frac{nr}{100})$

$= 10,000\left(1 + \frac{5 \times 10}{100}\right)$

$= 10,000\left(1 + \frac{50}{100}\right)$

$= 10,000\left(\frac{150}{100}\right)$

= 10,000 x 1.5 = 15,000

3) Find the simple interest and amount for Rs. 25,000 at 10% p. a for 26 weeks.

Ans: P = 25,000 n = 26/52, r = 10%

$$\text{Interest} = \frac{Pnr}{100} = \frac{25{,}000 \times \frac{26}{52} \times 10}{100}$$

$$= \frac{25{,}000 \times ½ \times 10}{100}$$

$$= \frac{25{,}000 \times 5}{100} = \underline{\underline{1250}}$$

$$\text{Amount at the end} = P\left(1 + \frac{nr}{100}\right)$$

$$= 25000\left(1 + \frac{\frac{26}{52} \times 10}{100}\right)$$

$$= 25000\left(1 + \frac{5}{100}\right)$$

$$= 25000\left(\frac{105}{100}\right)$$

$$= 25000 \times 1.05 = \underline{\underline{26250}}$$

4) Find the simple interest and amount for Rs. 50,000 at 7.5% p. a for 4 months.

Ans: P = 50,000, n = 4/12, r = 7.5%

$$\text{Simple Interest} = \frac{50{,}000 \times \frac{4}{12} \times 7.5}{100}$$

$$= \frac{50{,}000 \times 1/3 \times 7.5}{100}$$

$$= \frac{50{,}000 \times 2.5}{100} = \underline{\underline{1250}}$$

$$\text{Amount} = 5000\left(1 + \frac{\frac{4}{12} \times 7.5}{100}\right)$$

$$= 5000\left(1 + \frac{2.5}{100}\right)$$

$$= 5000\left(\frac{102.5}{100}\right)$$

$$= 5000 \times 1.025 = \underline{\underline{51250}}$$

5) Find the number of years in which a sum of money will double itself at 25% p. a, simple interest.

Ans: P = p, Amount = 2P, r = 25, n = ?

Amount = $P(1 + \frac{nr}{100})$

$2P = P(1 + \frac{nr}{100})$

i.e., $2 = (1 + \frac{nr}{100})$

$= 2 - 1 = \frac{nr}{100}$

$= 1 = \frac{nr}{100}$

nr = 100

r = 25, ∴ n = 4

number of years = 4

6) At what rate would a sum of money double in 20 years ?

Ans: P = p, A = 2p, n = 20, r = ?

Amount = $P(1 + \frac{nr}{100})$

$2P = P(1 + \frac{nr}{100})$

i.e., $2 = 1 + \frac{nr}{100}$

$= 2 - 1 = \frac{nr}{100}$

$= 1 = \frac{nr}{100}$

= nr = 100

n = 20, then r = 5

∴ Rate of interest = 5% per annum.

7) Find the number of years an amount of Rs. 8000 will take to become Rs. 12000 at 6% p. a. Simple interest.

Ans: P = 8000, A = 12000, r = 6, n = ?

Total interest 12000 – 8000 = 2000

Interest = $\frac{Pnr}{100}$

4000 = $\frac{8000\text{x n x }6}{100}$

4000 x 100 = 8000 x 6 x n

400000 = 48000n

48000n = 4,00,000

n = $\frac{400000}{48000}$ = 8.33 years

8) Find the rate of interest at which an amount of Rs. 12000 will become Rs. 15000 at the end of 10th year.

Ans: A = 15000, P = 12000, n = 1, r = ?

Total interest 15000 – 12000 = 3000

Interest = $\frac{Pnr}{100}$

3000 = $\frac{12000\text{x }10\text{ x r}}{100}$

3000 x 100 = 12000 x 10 x r

300000 = 120000r

r = $\frac{300000}{120000}$ = 2.5

Rate of interest = 2.5%

9) A certain sum amounts to Rs. 678 in 2 years and to Rs. 736.50 in 3-5 years find the rate of interest and principal amount.

Ans: Amount for 2 years = 678

“ 3-5 years = 736.50

Amount = P(1 +$\frac{nr}{100}$)

678 = P(1 +$\frac{2r}{100}$) ---------(1)

736.50 = P(1 +$\frac{3.5r}{100}$) --------(2)

Divide (1) by (2)

$= \frac{678}{736.50} = \frac{1+\frac{2r}{100}}{1+\frac{3.5r}{100}}$

$= \frac{678}{736.50} = \frac{100+2r}{100+3.5r}$

= 678(100 + 3.5r) = 736.50(100 + 2r)

= 67800 + 2373r = 73650 + 1473r

= 2373r – 1473r = 73650 – 67800

= 900r = 5850

= r = 5850/900 = 6.5

Substituting the value of r

$P(1+\frac{2r}{100}) = 678$

$P(1+\frac{2 \times 6.5}{100}) = 678$

$P(1+\frac{13}{100}) = 678$

$P(\frac{113}{100}) = 678$

P(1.13) = 678

P = 678/1.13 = 600

Rate of interest = 6.5%

Principal amount at the begining = 600

10) A person lends Rs. 1500, a part of it at 5% p.a. and the other part at 9% p.a. If he receives a total amount of interest of Rs. 162 at the end of 2 years. Find the amount lent at different rate of interest.

Ans: Let x is the Principal of 1st part

Then principal of 2nd part = 1500 – x

Total interest = 162

Interest = $\frac{Pnr}{100}$

Total interest = interest of 1st part and interest of 2nd part

$162 = \frac{x \times 2 \times 5}{100} + \frac{(1500 - X) \times 2 \times 9}{100}$

$= \frac{10x}{100} + \frac{(1500 - x) \times 18}{100} = 162$

$= \frac{10x + (27000 - 18x)}{100} = 162$

10x + (27000 – 18 x) = 162 x 100

10x – 18x = 16200 – 27000

-8x = -10800

8x = 10800

x = 10800/8 = 1350

Principal amount of 1st part = 1350

Principal amount of 2nd part = 150

**Compound Interest**

Compound interest means interest calculated on principal amount plus interest. Let ‘p’ be the principal ‘r’ be the rate of interest (compound) p.a., ‘n’ be the number of years then

$\text{Amount} = P\left(1 + \frac{r}{100}\right)^n$

Total interest = A – P

1) Find CI on Rs. 25200 for 2 years at 10% p.a compounded annually?

Ans: P = 25200, r = 10, n = 2

$A = P\left(1 + \frac{r}{100}\right)^n$

$= 25200\left(1 + \frac{10}{100}\right)^2$

$= 25200\left(\frac{110}{100}\right)^2$

$= 25200 \times (1.10)^2$

= 25200 x 1.21 = 30492

C1 = 30492 – 25200

= 5292

======

2) Find the Compound Interest Rs.10,000/- for 2½ years at 10% p.a..

Ans: P = 10,000 n = 2½ r = 10

Amount for 2 years = $p\left(1+\frac{r}{100}\right)^{n}$

$= 10,000\left(1+\frac{10}{100}\right)^{2}$

$= 10,000\left(\frac{110}{100}\right)^{2}$

$= 10,000 \times (1.1)^{2}$

= 10,000 x 1.21

= 12,100/-

Interest for 2 years = 2100

Interest for 6 months = $12100 \times \frac{10}{100} \times \frac{6}{12}$

= 605

Total interest for 2½ years = 2100 + 605

= 2,705/-
=======

3) X borrowed Rs.26,400/- from a bank to buy a scooter at the rate of 15% p.a. compounded yearly. What amount will be pay at the end of 2 years and 4 months to clear the loan.

Ans: p = 26,400/- r = 15

n = 2 years 4 months ($2\ {}^{1}/_{3}\ years$ )

Amount at the end of 2 years $= p\left(1+\frac{r}{100}\right)^{n}$

$= 26400\left(1+\frac{15}{100}\right)^{2}$

$= 26400\left(\frac{115}{100}\right)^{2}$

$= 26400\ (1.15)^{2}$

= 34,914

Interest for 4 months $= 34914 \times \frac{15}{100} \times \frac{4}{12}$

$= 1745.7$

Total amount at the end of 2 years and 4 months

ie 34914 + 1745.7 = 36659.7

==========

4) Mr. A borrowed Rs.20,000/- from a person, but he could not repay any amount in a period of 4 years. So the lender demanded as 26500 which is the rate of interest charged.

Ans: Here interest charged on compound

P = 20,000 n = 4 A = 26500 r = ?

$$A = p\left(1 + \frac{r}{100}\right)^n$$

$$26500 = 20000\left(1 + \frac{r}{100}\right)^4$$

$$\frac{26500}{20000} = \left(1 + \frac{r}{100}\right)^4$$

$$1.325 = \left(1 + \frac{r}{100}\right)^4$$

$$\log 1.325 = 4 \log\left(1 + \frac{r}{100}\right)$$

$$0.1222 = 4 \log\left(1 + \frac{r}{100}\right)$$

$$\log\left(1 + \frac{r}{100}\right) = \frac{0.1222}{4}$$

$$\log\left(1 + \frac{r}{100}\right) = 0.03055$$

Antilog 0.03055 = 1.073

$$\left(1 + \frac{r}{100}\right) = 1.073$$

$$\frac{r}{100} = 1.073 - 1$$

$$\frac{r}{100} = 0.073$$

$$r = 100 \times 0.073 = 7.3\%$$

=======

5) The population of a country increases every year by 2.4% of the population at the beginning of first year. In what time will be population double itself? Answer to the nearest year?

Ans: p = p A = 2p r = 2.4 n = ?

$$A = p\left(1 + \frac{r}{100}\right)^n$$

$$2p = p\left(1 + \frac{2.4}{100}\right)^n$$

$$2p = p\left(\frac{102.4}{100}\right)^n$$

$$2p = p\,(1.024)^n$$

$$p = (1.024)^n$$

log 2 = n log 1.024

0.3010 = n x 0.0103

$$n = \frac{0.3010}{0.0103} = 29.22 = 30$$

===

6) The population of a city increases every year by 1.8% of the population at the beginning of that year, in how many years will the total increase of population be 30%?

Ans: p = p A = 1.3p r = 1.8 n = ?

$$A = p\left(1 + \frac{r}{100}\right)^n$$

$$1.3\,p = p\left(1 + \frac{1.8}{100}\right)^n$$

$$1.3p = p\left(\frac{101.8}{100}\right)^n$$

$$1.3\,p = p\,(1.018)^n$$

$$1.3 = (1.018)^n$$

log 1.3 = n log 1.018

0.1139 = n x 0.0076

$$n = \frac{0.1139}{0.0076} = 14.987$$

= 15

=====

7) In a certain population, the annual birth and death rates per thousand are 39.4 and 19.4 respectively. Find the number of years in which population will be doubled assuming that there is no emigration or immigration?

Ans: p = p A = 2p

$r = \frac{39.4-19.4}{1000} \times 100 = 2\%$

r = 2 n = ?

$A = p\left(1+\frac{r}{100}\right)^n$

$2p = p\left(1+\frac{2}{100}\right)^n$

$2 = \left(1+\frac{2}{100}\right)^n$

$2 = p\,(1.02)^n$

log 2 = n log 1.02

0.3010 = n x 0.0086

$n = \frac{0.3010}{0.0086}$ = 35 years

==========

**COMPOUNDING HALF YEARLY OR QUARTERLY**

- When interest is compounded half yearly, then $r = {}^{r}/_{2}$ , n = 2n.
- When interest is compounded quarterly, then $r = {}^{r}/_{4}$ , n = 4n.
- When interest is compounded monthly, then $r = {}^{r}/_{12}$ , n = 12n.

1) Find the compound interest on Rs.50,000/- for 2 ½ years at 6% p.a. interest being compounded half yearly.

Ans: p = 50,000 n = 2 ½ x 2 = 5

$r = {}^{6}/_{2} = 3$

$\text{Amount} = 50{,}000\left(1+\frac{3}{100}\right)^5$

$= 50{,}000\left(\frac{103}{100}\right)^5$

$= 50{,}000\,(1.03)^5$ = 57964

C1 = 7964

========

2) Find the compound interest on Rs.60,000/- for 4 years, if interest is payable half yearly for due first 3 years at the rate of 8% p.a. and for the fourth year, the interest is being payable quarterly at the rate of 6% p.a.

Ans: Amount at in end of 3 years

n = 3 x 2 = 6, $r = \frac{8}{2} = 4$

p = 6,000

$= 6{,}000\left(1 + \frac{4}{100}\right)^6$

$= 6{,}000\left(\frac{104}{100}\right)^6$

$= 6{,}000\,(1.04)^6$

= 6,000 x 1.2653

= 7592

========

For last year

n = 1 x 4 = 4, $r = \frac{6}{2} = 1.5$, p = 7,592

Amount at the end of $4^{th}$ year

$= 7592\left(1 + \frac{1.5}{100}\right)^4$

$= 7592\,(1.015)^4$

= 7592 x 1.0613 = 8057

Interest = 8057 – 6000 = 2057

========

3) Find the effective rate of interest if interest is calculated at 10% p.a. half yearly?

Ans: Let p = 100, n = 1 x 2 = 2, $r = \frac{10}{2} = 5$

$A = p\left(1 + \frac{r}{100}\right)^n$

$= 100\left(1 + \frac{5}{100}\right)^2$

$= 100\left(\frac{105}{100}\right)^2$

= 100 x 1.1025 = 110.25

C 1 = 110.25 – 100 = 10.25

Effective rate = 10.25% p.a.

============

# MODULE IV

## MEANING AND DEFINITIONS OF STATISTICS

The word statistics is derived from the Latin word 'Status' or Italian word 'Statista' or German word 'Statistik' which means a Political State. It is termed as political state, since in early years, statics indicates a collection of facts about the people in the state for administration or political purpose.

Statistics has been defined either as a singular non or as a plural noun.

Definition of Statistics as Plural noun or as numerical facts:- According to Horace Secrist, 'Statistics are aggregates of facts affected to a marked extent by multiplicity of causes numerically expressed, enumerated or estimated according to a reasonable standard of accuracy, collected in a systematic manner for a predetermined purpose and placed in relation to each other'.

Definition of Statistics as a singular noun or as a method:- According to Seliman, " Statistics is the science which deals with the methods of collecting classifying, comparing and interpreting numerical data collected, to know some light on any sphere of enquiry".

**Characteristics of Statistics**

(1) Statistics show be aggregates of facts

(2) They should be affected to a marked extent by multiplicity of causes.

(3) They must be numerically expressed.

(4) They should be enumerated or estimated according to a reasonable standard of accuracy.

(5) They should be collected in a systematic manner.

(6) They should be collected for a predetermined purpose.

(7) They should be placed in relation to each other.

**Function of Statistics**

The following are the important functions of statistics:

1. It simplifies complexity:- Statistical methods make facts and figures easily understandable form. For this purpose Graphs and Diagrams, classification, averages etc are used.
2. It presents facts in a proper form:- Statistics presents facts in a precise and definite form.
3. It facilitates for comparison:- When date are presented in a simplified form, it is easy to compare date.

4. It facilitates for formulating policies:- Statistics helps for formulating policies for the companies, individuals, Govt. etc. it is possible only with the help of date presented in a suitable form.
5. It tests hypothesis:- Hypothesis is an important concept in research studies. Statistics provides various methods for testing the hypothesis. The important tests are Chi – square, Z-test, T-test and F-test.
6. It helps prediction or forecasting:- Statistical methods provide helpful means of forecasting future events.
7. It enlarges individual's knowledge:-When data are presented in a form of comparison, the individuals try to find out the reasons for the variations of two or more figures. It thereby helps to enlarge the individual's knowledge.
8. It measures the trend behavior:- Statistics helps for predicting the future with the help of present and past data. Hence plans, programs, and policies are formulated in advance with the help of statistical techniques.

Scope of Statistics or importance or utility of statistics.

The Scope of Statistics in various field are:

(1) Statistics in Business:- Statistics is most commonly used in business. It helps to take decision making of the business. The statistical data regarding the demand and supply of product can be collected and analyzed to take decisions. The company can also calculate the cost of production and then the selling price. The existing firms can also make a comparative study about their performance with the performance of others through statistical analysis.

(2) Statistics in Management:- Most of the managerial decisions are taken with the help of statistics. The important managerial activities like planning, directing and controlling are properly executed with the help of statistical data and statistical analysis. Statistical techniques can also be used for the payment of wages to the employees of the organization.

(3) Statistics in economics:- Statistical data and methods of statistical analysis render valuable assistance in the proper understanding of the economic problems and the formulation of economic policy.

(4) Statistics in banking and finance:- Banking and financial activities use statistics most commonly.

(5) Statistics in Administration:- The govt. frames polices on the basis of statistical information.

(6) Statistics in research:- Research work are undertaken with the help of statistics.

<u>Limitation of statistics</u>

(1) Statistics studies only numerical data

(2) Statistics does not study individual cases.

(3) Statistical result are true only an average.

(4) Statistics does not reveal the entire story of the problem.

(5) Statistics in only one of the methods of study a problem.

(6) Statistics can be misused.

<u>Statistical Enquires or Investigation</u>

Statistical Investigation is concerned with investigation of some problem with the help of statistical methods. It implies search for knowledge about some problems through statistical device.

Different stages in statistical enquiry are:

(1) Planning the enquiry

(2) Collection of data.

(3) Organization of data.

(4) Presentation of data.

(5) Analysis of data.

(6) Interpretation of data.

(1) Planning the enquiry:- The first step in statistical investigation is planning. The investigator should determine the objective and scope of the investigation. He should decide in advance about the type of enquiry to be conducted, source of information and the unit of measurement.

Object and scope:- The objective of the Statistical enquiry must be clearly defined. Once the objective of enquiry has been determined, the next step is to decide the scope of enquiry. It refers to the coverage of the enquiry.

Source of information:- After the purpose and scope have been defined, the next step is to decide about the sources of data. The sources of information may be either primary or secondary.

Types of enquiry:- Selection of type of enquiry depends on a number of factors like object and scope of enquiries, availability of time, money and facilities. Enquiries may be (1) census or sample (2) original or repetitive (3) direct or indirect (4) open or confidential (5) General or special purpose.

Statistical unit:- The unit of measurements which are applied in the collected data is called statistical unit. For example ton, gram, meter, hour etc.

Degree of accuracy:- The investigator has to decide about the degree of accuracy that he wants to attain. Degree of accuracy desired primarily depends up on the object of an enquiry.

Cost of plan:- An estimate of the cost of the enquiry must be prepaid before the commencement of enquiry.

(2) Collection of data:- Collection of data implies accounting and systematic recoding of the information gathered in a statistical investigation. Depending on the source, the collected

statistical data are classified under two categories namely primary data and secondary data.

(3) Organization of data:- Organization of data implies the arrangement and presentation of data in such a way that it becomes easy and convenient to use them. Classification and tabulation are the two stages of organizing data.

(4) Presentation of data:- They are numerous ways in which statistical data may be displayed. Graphs and diagrams are used for presenting the statistical data.

(5) Analysis data:- Analysis of data means critical examination of the data for studying characteristics of the object under study and for determining the pattern of relationship among the variables.

(6) Interpretation of data:- Interpretation refers to the technique of drawing inference from the collected facts and explaining the significance.

**Classification according to variables**

Data are classified on the basis of quantitative characteristics such as age, height, weight etc.

Geographical Classification:- Classified according to geographical differences.

Chronological Classification:- Classified according to period wise.

**Frequency Distribution**

A frequency distribution is an orderly arrangement of data classified according to the magnitude of observations. When data are grouped into classes of appropriate size indicating the number of observations in each class we get a frequency distribution.

**Components of frequency Distribution**

(1) Class and class interval

(2) Class limits

**Methods of classification**

(1) Classification according to attributes.

(2) Classification according to variables.

Classification according to attributes

Under this methods the data are classified on the basis of attributes. For example literacy, unemployment etc. are attributes.

Following are the classification under this method.

1. Simple classification
2. Manifold classification

In simple classification the data are divided on the basis of only one attributes.

In manifold classification the data are classified on the basis more than one attributes. For example population is divided on the basis of sex and literacy.

3. Class mark
4. Class boundaries
5. Magnitude of class interval
6. Class frequency.

**Tabulation**

Tabulation is an orderly arrangement of data in rows and columns. It is a moment of presentation of data.

**Objectives**

1. To simplify complex data
2. To facilitate comparison
3. To facilitate statistical analysis
4. To save time
5. To economies space

**Part of a table**

1. Table number
2. Title of the table
3. Caption -------- i.e. column headings
4. Sub -------------- i.e. row heading
5. Body
6. Head note
7. Foot note
8. Source data.

**Collection of data**

On the basis of source, data can be collected from primary and secondary source.

<u>Primary data</u>

Primary data are those collected by the investigator himself. May are original in character. May are truthful and suit for the purpose. But the collection is very expensive and time consuming.

Methods of collection of primary data

1. <u>Direct personal interview</u>:- In this method investigator collection the data personally. He was to meet the people for collecting the data. This method is suitable:
   a) When the area of investigation is limited
   b) When higher degree of accuracy is leaded.
   c) When the results of investigation to be kept confidential.

2. Indirect oral investigation:- Under this method, information are collected from third parties who are is touch with the facts under enquiry.
3. Schedules and Questionnaires methods:- Under this method, a list of questions called questionnaire is prepared and information are called from various sources. It is a printed list of questions to be filled by the informations. But schedule is filled by the enumerator.

**Essentials of a good questionnaire**

(1) The person conducting the survey much introduce himself.

(2) The number of questions should be kept to the minimum.

(3) The question should be as short as possible and simple.

(4) The questions must be arranged in logical order.

(5) The questions should be clear.

(6) Personal questions should be avoided.

(7) Questions should be in the nature of yes or no type.

(8) Questions must be of convenient size and easy to handle.

(9) Questions should be attractive.

(10) Instructions should be given for filing up the form.

Specimen of questionnaire.

**Secondary data**

Secondary data are those data which are collected by someone for this purpose. Secondary data are usually in the shape of finished product. The collection of secondary data is less expensive and less time consuming. Secondary data are collected from published and unpublished sources.

**Precautions to be taken before using secondary data**

(1) Suitability

(2) Adequacy

(3) Reliability

**Difference between Primary and Secondary data**

1. Primary data are original character. But secondary data are not original, they are collected by somebody else.
2. Primary data are in the shape of raw material. But secondary data are in the shape of finished product.
3. Collection of primary data is expanse and time consuming. But collection of secondary data is less expensive and less time consuming.
4. Primary data will be usually adequate and suitable. But secondary data need not be adequate and suitable for the purpose.

**Sampling**

Sampling is the process obtaining information about an entire population by examining only a part of it. It is the examination of the regenerative items and conclusion of draw for all items coming in that group.

**Methods of sampling or techniques of sampling**

1. Probability sampling or random sampling
2. Non probability sampling

**Probability sampling**

Under this method, each items has an equal chance for being selected.

Following are the random sampling.

**(1) Simple random sampling**

A simple random sample is a sample selected from a population in such a way that every item of the population has an equal chance of being selected. The selection depends on chance. Eg. Lottery methods.

**(2) Systematic sampling**

This method is popularly used in those cases where complete list of the population from which sample is to be drawn is available. Under this method the items in the population are included in intervals of magnitude K. From every interval select an item by simple random sample method.

**(3) Cluster sampling**

Cluster sampling consists in forming suitable clusters of units. All the units is the sample of clusters selected are surveyed.

**(4) Quota sampling**

In this method each investigator engaged in the collection of data is assigned a quota for investigation.

**(5) Multi stage sampling**

This is a sampling procedure carried but in several stages. In multistage sampling, firstly units selected by suitable methods of sampling. From among the selected units, sample is drawn by some suitable methods. Further stages are added to arrive at a sample of the desired units.

**Non probability sampling**

1. Judgment sampling:- Under this sampling investigator exercise this discretion in the mater of selecting the items that are to be included in the sample.

2. Convenience Sampling:- Convenience sampling is one in which a sample is obtained by selecting such units of the universe which may be conveniently located.

**Organization of data**

Organizing data mean, the arrangement and presentation of data. Classification and tabulation are the two stages of organizing data.

**Classification**

The process of arranging data in groups or classes according to similarities called classification.

**Objects of classification**

1. To simplify the complexity of data.
2. To bring out the points of similarity of the various items.
3. To facilitate comparison.
4. To bring out relationship.
5. To provide basis for tabulation.

**Graphs and Diagrams**

Graphs and diagrams is one of the statistical methods which simplifies the complexity of quantitative data and make them easily understandable.

**Importance of Diagrams & Graphs**

1. Attract common people
2. Presenting quantitative facts in simple.
3. They have a great memorizing effect.
4. They facilitate comparison of data.
5. Save time in understanding data.
6. Facts can be a understood without mathematical calculations.

**Limitations**

1. They can present only approximate values.
2. They can represent only limited amount of information.
3. They can be misused very easily.
4. They are not capable of further mathematical treatment.
5. They are generally useful for comparison purpose only.

**General rules for constructing Diagrams**

1. Title
2. Proportion between width and height.
3. Selection of scale
4. Foot note
5. Index
6. Neatness and cleanliness
7. Simplicity
8. Attractiveness

**Types of Diagrams**

1. Dimensional Diagrams
2. Cartograms
3. Pictograms

**Dimensional Diagrams**

Dimensional Diagrams are those diagrams which show information in terms of length, height, area or volume. They are one dimensional two dimensional or three dimensional.

**One Dimensional Diagram**

In one dimensional diagram the height will represent the magnitude of observations. Must commonly used one dimensional diagrams are line diagram and Bar diagram.

**Line Diagram**

Line diagrams are one dimensional diagrams. They are drawn to represent values of a variable.

Ex. Draw a line diagram to the following data.

| | | | | | |
|---|---|---|---|---|---|
| Country: | A | B | C | D | E |
| Population: (in million) | 10 | 5 | 15 | 13 | 12 |

15

10

Population (in million)

5

A B C D E

Column

**Bar Diagrams**

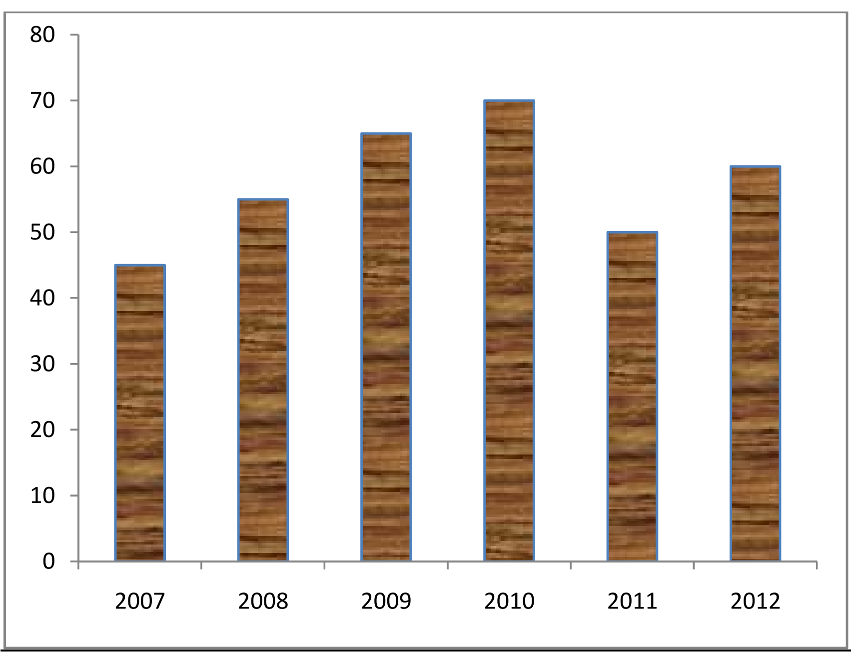
In a bar diagram only the length is considered. The width of the bar is not given any importance.

Following are the important types of bar diagrams.

(1) Simple bar diagram

Simple bar diagram represents only one variable. For example height, weight, etc.

| | | | | | | |
|---|---|---|---|---|---|---|
| Year: | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
| Sales In '000') | 45 | 55 | 65 | 70 | 50 | 60 |

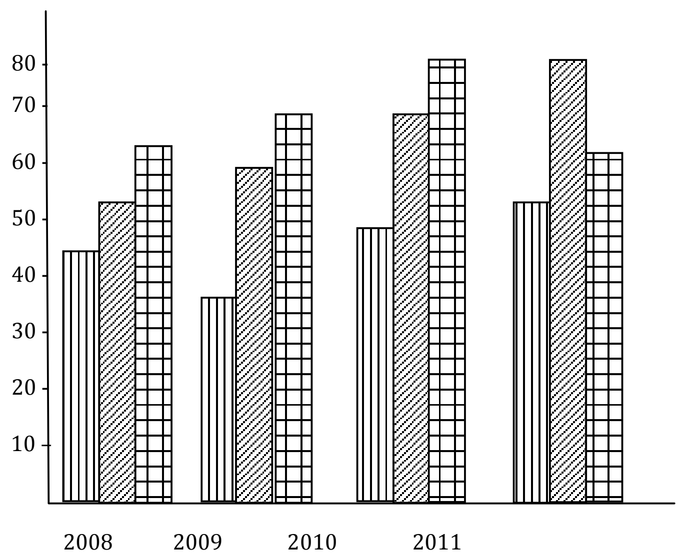


## 2) Multiple Bar Diagram

Two or more interrelated data are represented in a multiple bar diagram. In order to identity the data, the bars should be differentiated with colors or shades.

Eg:- From the following data draw a suitable diagram.

| Year | Production (in units) | | |
|---|---|---|---|
| | A | B | C |
| 2008 | 45 | 55 | 65 |
| 2009 | 35 | 60 | 70 |
| 2010 | 50 | 70 | 80 |
| 2011 | 55 | 80 | 60 |

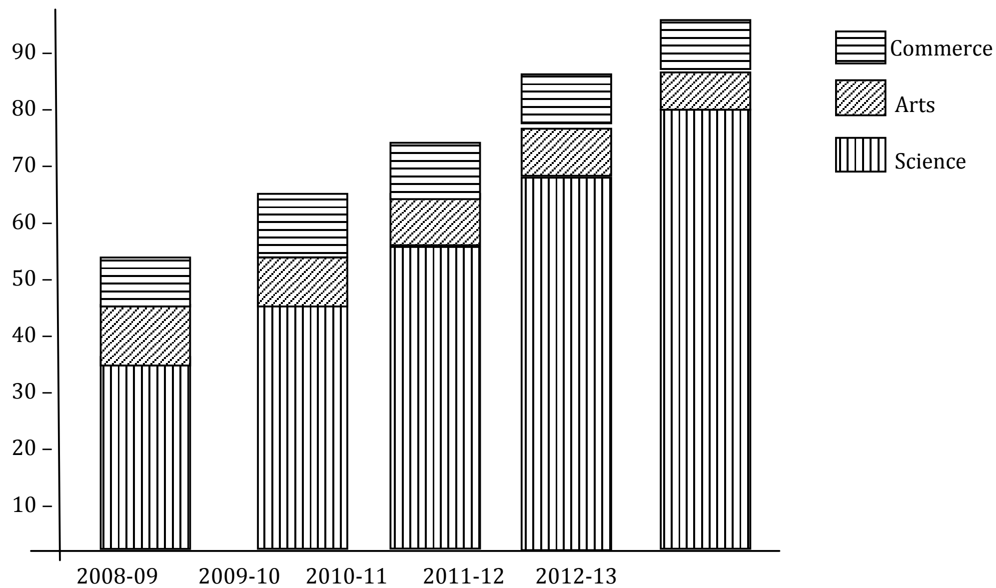



## 3) Sub Divided Bar Diagram

In the sub divided bar diagram each bar is subdivided into two or more parts. Each part may explain different characters.

Eg:- The number of students in Calicut University are as follows: Represent the date by suitable diagram

| Year | Commerce | Arts | Science | Total |
|---|---|---|---|---|
| 2008-09 | 35000 | 10000 | 9000 | 54000 |
| 2009-10 | 45000 | 9000 | 90000 | 64000 |
| 2010-11 | 55000 | 7000 | 8000 | 69000 |
| 2011-12 | 70000 | 5000 | 7000 | 82000 |
| 2012-13 | 80000 | 4000 | 6000 | 90000 |




## 4) Percentage Bar Diagrams

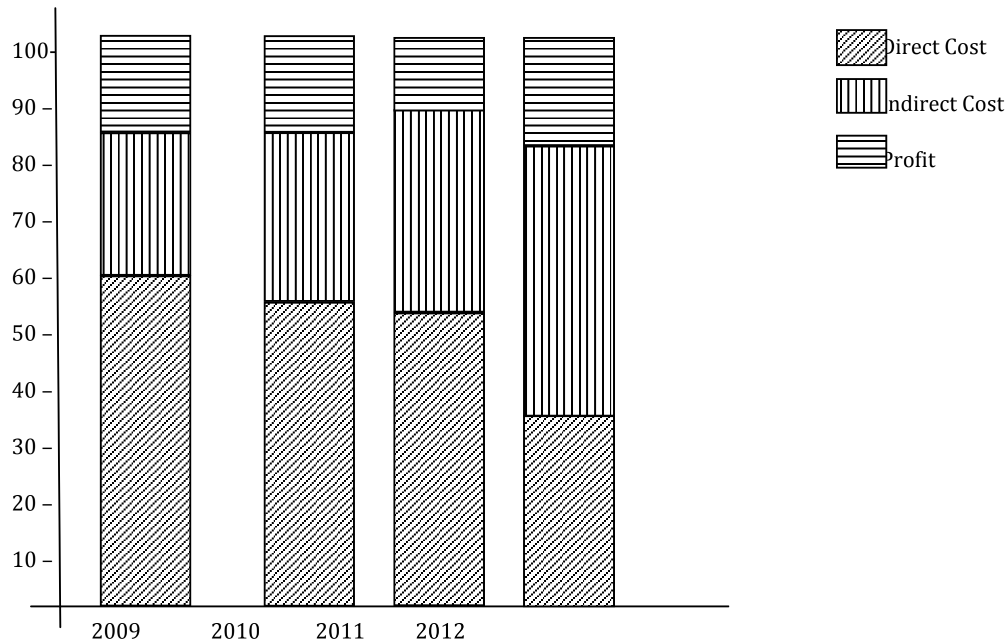
In percentage bar diagram the length of all the base are equal ie each bar represent 100 percent. The component parts are expressed as percentage to the whole.

Eg:- Prepare a subdivided bar diagram on the percentage basis.

| Year | Direct Cost Rs | Indirect Cost Rs | Profit Rs | Sales Rs |
|---|---|---|---|---|
| 2009 | 35 | 15 | 10 | 60 |
| 2010 | 40 | 20 | 12 | 72 |
| 2011 | 32 | 22 | 8 | 62 |
| 2012 | 25 | 35 | 15 | 75 |

Answer

| Year | Direct Cost in % | Indirect Cost in % | Profit in % | Sales |
|---|---|---|---|---|
| 2009 | 58 | 25 | 17 | 100 |
| 2010 | 55 | 28 | 17 | 100 |
| 2011 | 52 | 35 | 13 | 100 |
| 2012 | 33 | 47 | 20 | 100 |




**Two Dimensional Diagram**

In two dimensional diagram the length as well as width have to be considered. The most commonly used two dimensional diagrams is pie diagram, Rectangles, Squares, Circles etc are also two dimensional diagrams.

**Pie Diagrams**

Pie diagrams are used when the aggregate and their divisions are to be shown together. The aggregate is shown by means of a circle and divisions by the sectors of the circle. For example, the selling price of a product can be divided into various segments like factory cost, administrative cost, selling cost and profit. These segments are converted into percentage in order to represent in the pie diagram.

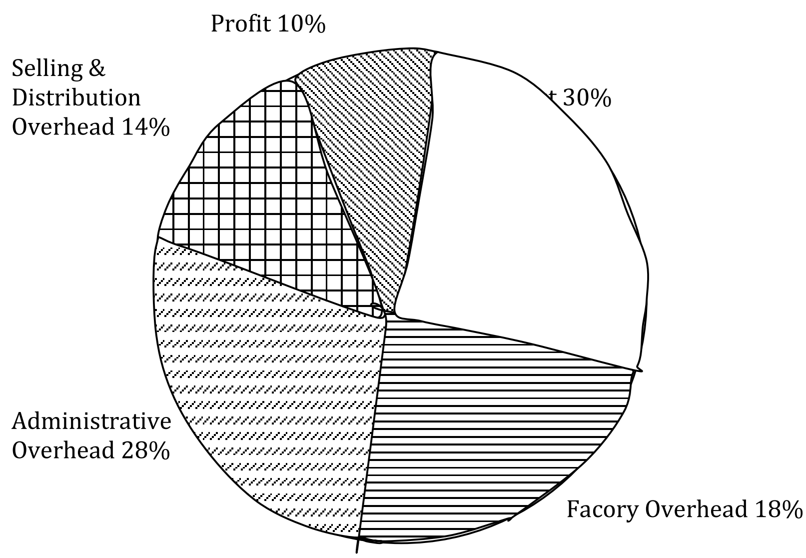
In order to prepare the pie diagram, each percentage outlay must be multiplied by 3.6, since the pie diagram contain $360^o$ scale.

Eg:- Draw a pie diagram from the following data

| | |
|---|---|
| Prime Cost | 30% |
| Factory over Head | 18% |
| Administrative overhead | 28% |
| Selling & Distribution overhead | 14% |
| Profit | 10% |

Ans:

| | | |
|---|---|---|
| Prime Cost | 30 | 108° |
| Factory over Head | 18 | 65° |
| Administrative overhead | 28 | 101° |
| Selling & Distribution overhead | 14 | 50° |
| Profit | 10 | 36° |
| | 100 | 360 |




## Three Dimensional Diagrams

Three dimensional diagrams are prepared in the form of cubes, spheres, cylinders etc. In these diagrams width, length and breadth are important.

## Cartograms

Cartograms means the presentation of data in a geographical basis. It is otherwise called as statistical maps. The quantities on the map may be shown through shades, dots or colours etc.

## Pictograms

Under the pictograms, data are represented in the form of a appropriate pictures most suited for the data.

# GRAPHS

## Types of Graphs

(1) Graphs of Frequency Distribution

(2) Graphs of Time Series

## Graphs of Frequency Distribution

A frequency distribution can be presented graphically in any of the following ways:

(1) Histogram

(2) Frequency Polygon

(3) Frequency Curves

(4) Ogive or cumulative frequency curves.

## Histogram

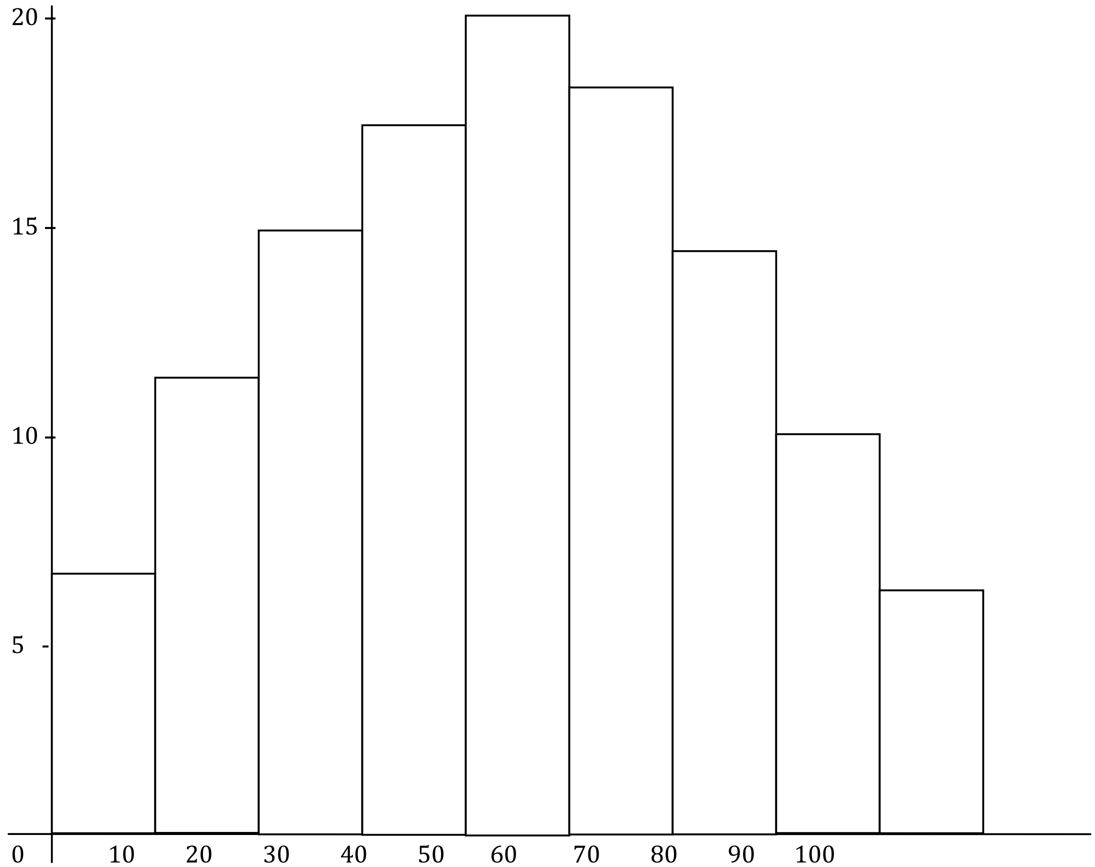
A histogram is a graph of frequency distributions. A histogram consists of bars erected upon the class interval columns.

While constructing histogram, the variable is always taken on the x-axis and the frequency on the y-axis. The width of the bars in the histogram will be proportional to the class interval.

## Histogram for frequency Distribution having equal Class interval

1) Draw a histogram from the following information

| Marks | No. of Students |
|---|---|
| 0-10 | 7 |
| 10-20 | 12 |
| 20-30 | 15 |
| 30-40 | 17 |
| 40-50 | 20 |
| 60-70 | 14 |
| 70-80 | 10 |
| 80-90 | 4 |




**Histogram for unequal Class Interval**

Unequal class intervals must be corrected.

$$\text{Unequal class intervals} = \frac{\text{Frequency unequal class intervals}}{\text{width of the unequal class intervals}} \times \text{width of the lowest class interval}$$

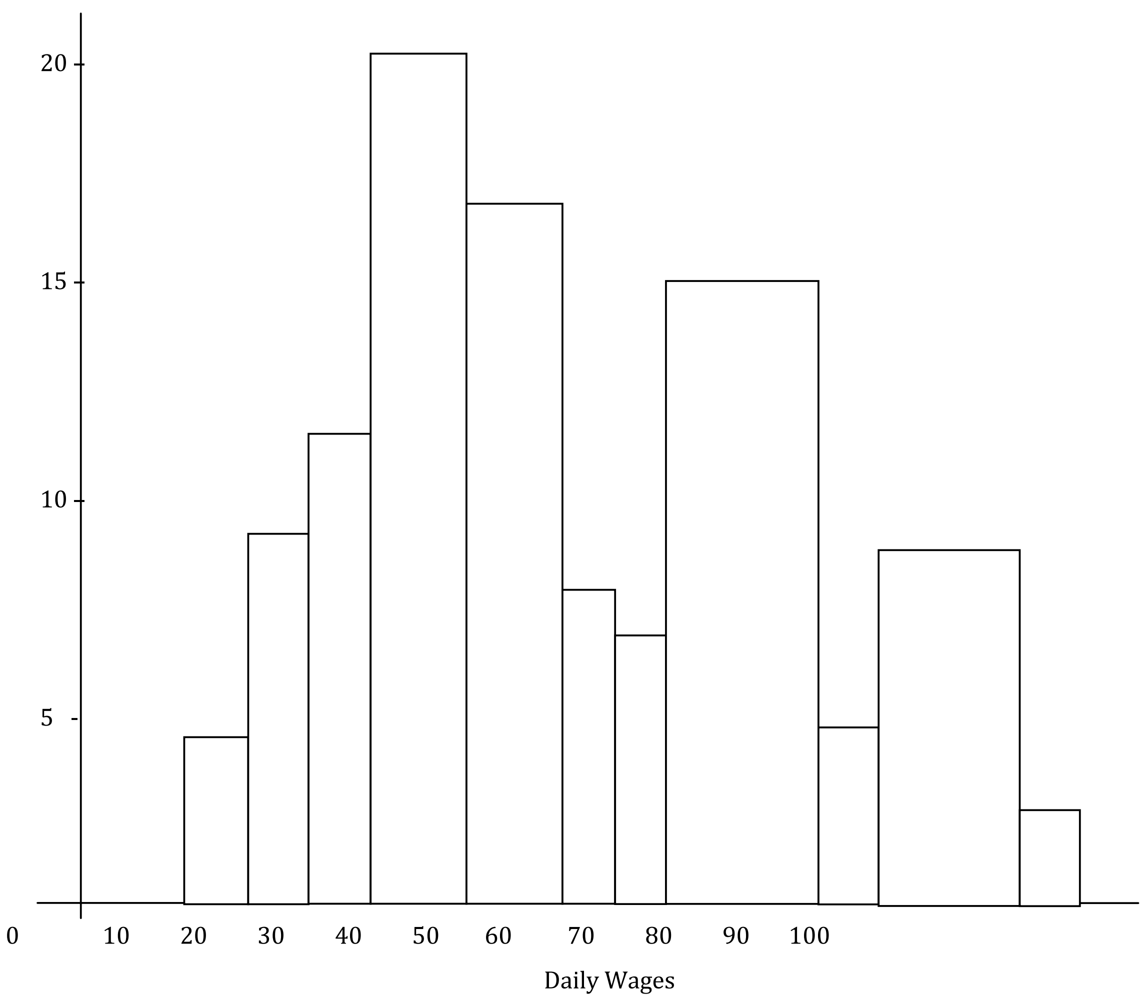
Draw a histogram from the following data

| Daily wages | No. of workers |
|---|---|
| 15-20 | 4 |
| 20-25 | 9 |
| 25-30 | 12 |
| 30-40 | 20 |
| 40-50 | 16 |
| 50-55 | 7 |
| 55-60 | 6 |
| 60-75 | 15 |
| 75-80 | 4 |
| 80-95 | 9 |
| 95-100 | 2 |

*Answer:*

Calculation of Frequency Density

| Daily wages | No. of workers | Frequency Density |
|---|---|---|
| 15-20 | 4 | 4 |
| 20-25 | 9 | 9 |
| 25-30 | 12 | 12 |
| 30-40 | 20 | 10 |
| 40-50 | 16 | 8 |
| 50-55 | 7 | 7 |
| 55-60 | 6 | 6 |
| 60-75 | 15 | 5 |
| 75-80 | 4 | 4 |
| 80-95 | 9 | 3 |
| 95-100 | 2 | 2 |



**Frequency Polygon**

It is a curve instead of bars. There are two methods for constructing frequency polygon. First, histogram should be drawn and mark mid point of upper side of each bar and join such joints by a curve.

In the second method, first of all plot the frequencies corresponding to midpoints of various class intervals. Then join all the plotted points to get the frequency polygon curve.

**3) Ogive or Cumulative Frequency Curve**

A frequency distribution when cumulated, we get cumulative frequency distribution and curve drawn is known as ogive. An ogive can either less than ogive or more than ogive. Less than ogive curve is drawn on the basis of less than cumulative frequency distribution and more than ogive is drawn on the basis of more than cumulative frequency distribution.

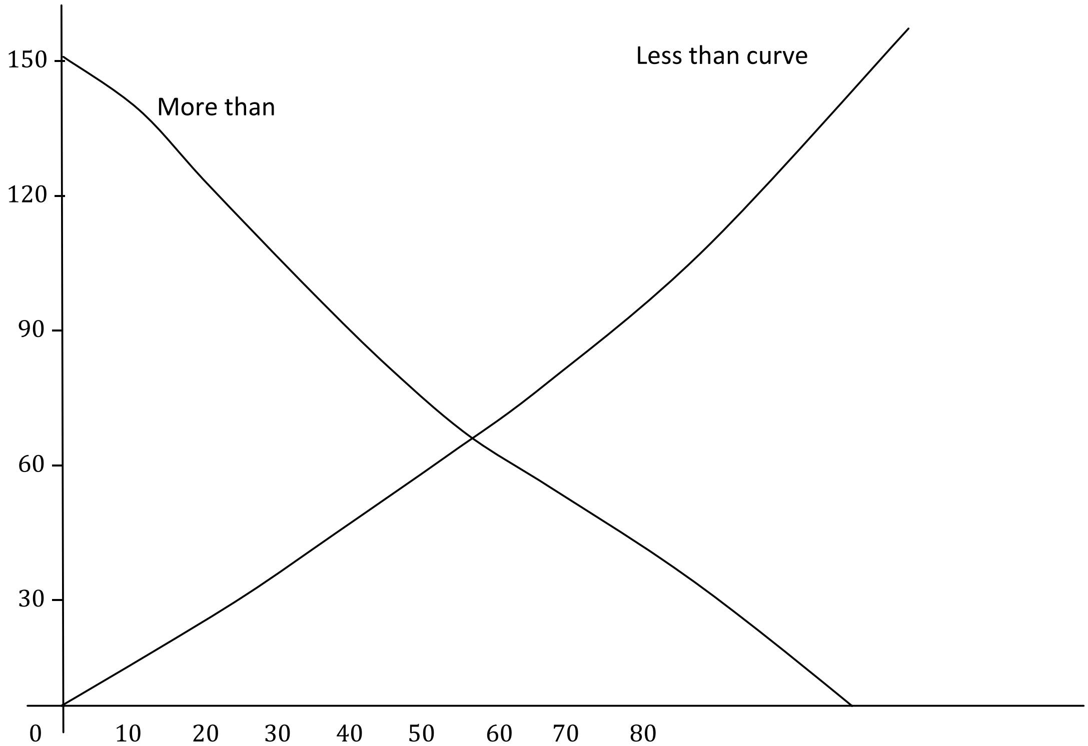
*Example :-*

From the following data drawn less than and more than ogives

| Marks : | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|---|---|---|
| No. of Students | 10 | 20 | 35 | 30 | 20 | 15 | 10 | 10 |

**Answer :**

| **Less than CF** | **F** | **More than CF** | **F** |
|---|---|---|---|
| Less than 0 | 0 | More than 0 | 150 |
| Less than 10 | 10 | More than 10 | 140 |
| Less than 20 | 30 | More than 20 | 120 |
| Less than 30 | 65 | More than 30 | 95 |
| Less than 40 | 95 | More than 40 | 55 |
| Less than 50 | 125 | More than 50 | 35 |
| Less than 60 | 130 | More than 60 | 20 |
| Less than 70 | 140 | More than 70 | 10 |
| Less than 80 | 150 | More than 80 | 0 |




**Measures of central tendency or Averages**

An average is a single value that represents a group of values. It represents the whole series and conveys general idea of the whole group. Characteristics of a good average or Requisites or Essentials properties of average

(1) Clearly defined
(2) Easy to understand
(3) Simple to compute
(4) Based on all items
(5) Not be unduly affected by extreme observations.
(6) Capable of further algebraic treatment
(7) Sampling stability.

**Types of averages**

1) Arithmetic Mean
2) Median
3) Mode
4) Geometric mean
5) Harmonic Mean

## Arithmetic Mean (AM)

It is the value obtained by adding together all the items and by dividing the total number of items.

Arithmetic mean may either be

(1) Simple arithmetic Mean or

(2) Weighted arithmetic Mean

### Simple Arithmetic Mean

It is the mean of items which give equal importance to all items.

It is denoted by $\bar{x}$

$$\bar{x} = \frac{\sum x}{N}$$

Where = Sum of given variables

N = Number of items

### Calculation of Arithmetic Mean

**(a) Individual Series :-**

(i) Direct Method

$$\bar{x} = \frac{\sum x}{N}$$

(ii) Short Cut Method

$$\bar{x} = A + \frac{\sum d}{n}$$

A = Assumed mean

D = X – A

n = total number of items

**(b) Discrete Series**

(i) Direct Method

$$\bar{x} = \frac{\sum fx}{N}$$

(ii) Short Cut method

$$\bar{x} = A + \frac{\sum fd}{N}$$

d = X – A

(iii) Step deviation method

$$\bar{x} = A + \frac{\sum fd'}{N} \times C$$

d' = $\frac{X-A}{C}$

c = common factor

**(c) Continuous Series**

(i) Direct method

$\bar{x} = \frac{\sum fm}{N}$

m = midpoint of X

N = Total frequency

(ii) Short cut method

$\bar{X}$ = A + $\frac{\sum fd}{N}$

d = m – A

(iii) Step deviation method:

$\bar{X}$ = A + $\frac{\sum fd'}{N}$ X C

d' = $\frac{m-A}{C}$

C = Common factor or class interval

*Practical Problems*

1) Calculate A.M. of the weight of 10 students in a Class

| Sl. No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Weight in Kg | 42 | 56 | 49 | 50 | 49 | 53 | 52 | 48 | 47 | 54 |

Ans: This is an individual series.

$\bar{X} = \frac{\sum X}{n}$

Σx = 42+56+49+50+49+53+52+48+47+54

= 500

n = 10

$\bar{x}$ = $\frac{500}{10}$ = 50Kg.

===========

2) Calculate mean from the following data.

| Marks | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of students | 3 | 8 | 12 | 9 | 4 | 7 | 15 | 5 | 10 | 7 |

Ans:

| Marks<br>x | No. of students<br>f | d<br>(x – 55) | d’ | fd’ |
|---|---|---|---|---|
| 25 | 3 | -30 | -6 | -18 |
| 30 | 8 | -25 | -5 | -40 |
| 35 | 12 | -20 | -4 | -48 |
| 40 | 9 | -15 | -3 | -27 |
| 45 | 4 | -10 | -2 | -8 |
| 50 | 7 | -5 | -1 | -7 |
| 55 | 15 | 0 | 0 | 0 |
| 60 | 5 | 5 | 1 | 15 |
| 65 | 10 | 10 | 2 | 20 |
| 70 | 7 | 15 | 3 | 21 |
| | 80 | | | -120 |

$\bar{X} = A + \frac{\sum fd'}{N} X\, C$

$\bar{X} = 55 + \frac{-102}{80} X\, 5$

$= 55 + \frac{-510}{80}$

= 55 + -6.375

= 48.625

==========

3. Calculate Arithmatic Mean

| Production in tons | No. of factories |
|---|---|
| 10 – 20 | 5 |
| 20 – 30 | 4 |
| 30 – 40 | 7 |
| 40 - 50 | 12 |
| 50 – 60 | 10 |
| 60 – 70 | 8 |
| 70 – 80 | 4 |

Ans:

| X | f | m | fm |
|---|---|---|---|
| 10 – 20 | 5 | 15 | 75 |
| 20 – 30 | 4 | 25 | 100 |
| 30 – 40 | 7 | 35 | 245 |
| 40 - 50 | 12 | 45 | 540 |
| 50 – 60 | 10 | 55 | 550 |
| 60 – 70 | 8 | 65 | 520 |
| 70 – 80 | 4 | 75 | 300 |
| | 80 | | 2330 |

$\bar{X} = \frac{\sum fm}{N} = \frac{2330}{50} = 46.6$

==========

4. Following are the data related with the production of a product during January in 100 factories

| Production in tons | No. of factories |
|---|---|
| 0 – 100 | 7 |
| 100 – 200 | 15 |
| 200 – 300 | 10 |
| 300 – 400 | 9 |
| 400 – 500 | 10 |
| 500 – 600 | 12 |
| 600 - 700 | 8 |
| 700 – 800 | 13 |
| 800 – 900 | 9 |
| 900 - 1000 | 7 |

Ans:

| x | f | m | d (m – A) | d’ | fd’ |
|---|---|---|---|---|---|
| 0 – 100 | 7 | 50 | -500 | -5 | -35 |
| 100 – 200 | 15 | 100 | -400 | -4 | -60 |
| 200 – 300 | 10 | 250 | -300 | -3 | -30 |
| 300 – 400 | 9 | 350 | -200 | -2 | -18 |
| 400 – 500 | 10 | 450 | -100 | -1 | -10 |
| 500 – 600 | 12 | 550 | 0 | 0 | 0 |
| 600 - 700 | 8 | 650 | 100 | 1 | 8 |
| 700 – 800 | 13 | 750 | 200 | 2 | 26 |
| 800 – 900 | 9 | 850 | 300 | 3 | 27 |
| 900 - 1000 | 7 | 950 | 400 | 4 | 28 |
| | 100 | | | | -64 |

$\bar{X}$ = A + $\frac{\sum fd'}{N}$ X C

$\bar{X}$ = 550 + $\frac{-64}{100}$ X 100

= 486

========

**<u>Calculation of Arithmatic Mean for open end classes</u>**

If the lower limit of the first class and upper limit of the last class are not known, it is called open end classes.

1. Calculate A.M.

| | |
|---|---|
| Below 10 | 5 |
| 10 – 20 | 12 |
| 20 – 30 | 14 |
| 30 – 40 | 10 |
| Above 40 | 8 |

Ans:

| X | f | m | fm |
|---|---|---|---|
| 0 – 10 | 5 | 5 | 25 |
| 10 – 20 | 12 | 15 | 180 |
| 20 – 30 | 14 | 25 | 350 |
| 30 – 40 | 10 | 35 | 350 |
| 40 - 50 | 8 | 45 | 360 |
| | 49 | | 1265 |

$\bar{X} = \frac{\sum fm}{N} = \frac{1265}{49} = 25.82$

==========

# MODULE - V

## Weighted Mean

Weighted means are obtained by taking in to account of weights. Each value is multiplied by its weight and total is divided by the total weight to get weighted mean.

$$\bar{x}_W = \frac{\sum wx}{\sum w}$$

$\bar{x}_W$ = weighted A.M.

w = weight

x = given variable

## Median

Median is the middle value of the series. When the series are arranged in the ascending order or descending order Median is a positional average.

## Calculation of Median

## Individual series

Firstly arrange the series.

Median = Size of $\left(\frac{n+1}{2}\right)^{th}$ item.

## Discrete series

Median = Size of $\left(\frac{n+1}{2}\right)^{th}$ item.

## Continuous series

Median Class = $\frac{N}{2}$

Median = $L_1 + \frac{N/_2 - c.f}{f} \times C$

$L_1$ = Lowerlimit of median class

c.f = culmulative frequency of preceding median class

f = frequency of median class

C = Class interval

1) Find the median for the following data

4, 25, 45, 15, 26, 35, 55, 28, 48

***Answer :***

4, 15, 21, 25, 26, 28, 35, 45, 48, 55

Median = $\left(\frac{N+1}{2}\right)$ $th$ item

$\left(\frac{9+1}{2}\right)$ $th$item =$5^{th}$ item

Median = 28

2) Calculate median

25, 35, 15, 18, 17, 36, 28, 24, 22, 26

***Answer*** :

15, 17, 18, 22, 24, 25, 26, 28, 35, 36

Median = $\left(\frac{N+1}{2}\right)^{th}$ item

$\left(\frac{10+1}{2}\right)^{th}$item

= 5.5 item

Median = $\frac{5^{th} item + 6^{th} item}{2}$

$\frac{24 + 25}{2}$ = 24.5

3) Calculate median

| | | | | | | |
|---|---|---|---|---|---|---|
| Size : | 5 | 8 | 10 | 15 | 20 | 25 |
| Frequency : | 3 | 12 | 8 | 7 | 5 | 4 |

**Answer :**

| Size | Frequency | Cf |
|---|---|---|
| 5 | 3 | 3 |
| 8 | 12 | 15 |
| 10 | 8 | 23 |
| 15 | 7 | 30 |
| 20 | 5 | 35 |
| 25 | 4 | 39 |

Median = $\left(\frac{N+1}{2}\right)^{th}$ item

$\left(\frac{39+1}{2}\right)^{th}$ item $=20^{th}$ item

Median = 10

4) Find median from the following :

| **Marks** | **No. of students** |
|---|---|
| 0-5 | 29 |
| 10-15 | 195 |
| 15-20 | 241 |
| 20-25 | 117 |
| 25-30 | 52 |
| 30-35 | 10 |
| 35-40 | 6 |
| 40-45 | 2 |

**Answer :**

| **Marks** | **f** | **c.f** |
|---|---|---|
| 0-5 | 29 | 29 |
| 5-10 | 195 | 227 |
| 10-15 | 241 | 465 |
| 15-20 | 117 | 582 |
| 20-25 | 52 | 634 |
| 25-30 | 10 | 644 |
| 30-35 | 6 | 650 |
| 35-40 | 3 | 653 |
| 40-45 | 3 | 656 |
| | -------- 656 === | |

Median class = $N/2 = \frac{656}{2} = 328^{th}$ item

Median = $L_1 + \frac{N/2 - cf}{f} \times$ C

$= 10 + \frac{328 - 224}{241} \times 5$

= 12.2

===

## Mode

Mode is the value of item of series which occurs most frequently.

### Mode in individual series

In the case of individual series, the value which occurs more number of times is mode.

When no items appear more number of times than others, then mode is the ill defined. In this case :

Mode = 3 median – 2 mean

## Mode in discrete series

In the case of discrete series, the value having highest frequency is taken as mode.

## Mode in continuous series

Mode lies in the class having the highest frequency.

$$\text{Mode} = l_1 + \frac{(f_1 - f_0)\times C}{2f_1 - f_0 - f_2}$$

$l_1$= lower limit of the model class

$f_1$= frequency of the model class

$f_0$,$f_1$= frequency of class preceding and succeeding modal class.

1) Find mode

1, 2, 5, 6, 7, 3, 4, 8, 2, 5, 4, 5

***Answer :***

Mode = 5

==

2) Find mode

4, 2, 6, 3, 8, 7, 9, 1

***Answer***

Mode is ill defined

Mode = 3 median – 2 mean

$\bar{x} = \frac{\sum x}{n} = \frac{40}{8} = 5$

Median : 1, 2, 3, 4, 6, 7, 9

Median = $\frac{N+1^{th}}{2}$item = $\frac{8+1}{2}$ = 4.5

Median = $\frac{4^{th}5^{th}item}{2}$ = $\frac{4+6}{10}$ = 5

Mode = 3 × 5 – 2 × 5 = 5

==

3) Find mode

| Size : | 5 | 8 | 10 | 12 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|
| Frequency: | 3 | 7 | 2 | 9 | 5 | 6 | 2 |

Mode = 2, since 12 has the highest frequency

4) Calculate mode

| Size : | 0-5 | 5-10 | 10-15 | 15-20 | 20-25 | 25-30 |
|---|---|---|---|---|---|---|
| Frequency: | 20 | 24 | 32 | 28 | 20 | 26 |

***Answer***

| **Size** | **Frequency** | |
|---|---|---|
| 0-5 | 20 | |
| 5-10 | 24 | |
| 10-15 | 32 | ---- -Models class |
| 15-20 | 28 | |
| 20-25 | 20 | |
| 25-30 | 26 | |

Mode = $l_1 + \frac{(f_1 - f_0)\times C}{2f_1 - f_0 - f_2}$

$= 10 + \frac{(32-24)\times 5}{2\times 32 - 24 - 28}$

$= 10 + \frac{40}{12}$

= 13.3

===

5) Calculate mean, median and mode

| Marks | No. of students |
|---|---|
| Less than 10 | 4 |
| Less than 20 | 9 |
| Less than 30 | 15 |
| Less than 40 | 18 |
| Less than 50 | 26 |
| Less than 60 | 30 |
| Less than 70 | 38 |
| Less than 80 | 50 |
| Less than 90 | 54 |
| Less than 100 | 55 |

**Answer :**

| Marks | Frequency | M | fm | c.f. |
|---|---|---|---|---|
| 0-10 | 4 | 5 | 20 | 4 |
| 10-20 | 5 | 15 | 75 | 9 |
| 20-30 | 6 | 25 | 150 | 15 |
| 30-40 | 3 | 35 | 105 | 18 |
| 40-50 | 8 | 45 | 360 | 26 |
| 50-60 | 4 | 55 | 220 | 30 |
| 60-70 | 8 | 65 | 520 | 38 |
| 70-80 | 12 | 75 | 900 | 50 |
| 80-90 | 4 | 85 | 340 | 54 |
| 90-100 | 1 | 95 | 95 | 55 |

Mean

$$\bar{x} = \frac{\sum fm}{N}$$

$$= \frac{2785}{55}$$

$$= 50.63$$

====

Median

$$= \frac{N^{th}}{2} \text{ item}$$

$$= \frac{55^{th}}{2} \text{ item}$$

$$= 27.5^{th}\text{item}$$

$$= l_1 + \frac{N/2 - c.f}{f} \times C$$

$$= 50 + \frac{27.5-26}{4} \times 10$$

$$= 50 + \frac{1.5}{4} \times 10$$

$$= 73.33$$

===

6) Calculate mean, median and mode

| Marks | No. of students |
|---|---|
| More than 0 | 80 |
| More than 10 | 77 |
| More than 20 | 72 |
| More than 30 | 65 |
| More than 40 | 55 |
| More than 50 | 43 |
| More than 60 | 28 |
| More than 70 | 16 |
| More than 80 | 10 |
| More than 90 | 8 |

**Answer**

| X | f | m | fm | c.f |
|---|---|---|---|---|
| 0-10 | 3 | 5 | 15 | 3 |
| 10-20 | 5 | 15 | 75 | 8 |
| 20-30 | 7 | 25 | 175 | 15 |
| 30-40 | 10 | 35 | 350 | 25 |
| 40-50 | 12 | 45 | 540 | 37 |
| 50-60 | 15 | 55 | 825 | 52 |
| 60-70 | 12 | 65 | 780 | 64 |
| 70-80 | 6 | 75 | 450 | 70 |
| 80-90 | 2 | 85 | 170 | 72 |
| 90-100 | 8 | 95 | 760 | 80 |

Mean

$\bar{X} = \frac{\sum fm}{N}$

$= \frac{4140}{8}$

$= 51.5$

===

Median $= {}^{80}/_{2}{}^{th}$ item

$= 40^{th}$ item

$= l_1 + \frac{{}^{N}/_{2} - c.f}{f} \times C$

$= 50 + \frac{40-37}{15} \times 10$

$= 50 + \frac{3}{15} \times 10$

$= 52$

==

Mode

$= l_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times C$

$= 50 + \frac{15-12}{2\times15-12-12} \times 10$

$= 50 + \frac{3}{30-12-12} \times 10$

$= 50 + \frac{3}{6}\times 10$

$= 55$
==

## Geometric Mean

Geometric mean is defined as the $n^{th}$ root of the product of those in values.

$$G.m = Antilog\left(\frac{\sum log x}{n}\right)$$

## G.M in Individual series

$$G.M = Antilog\left(\frac{\sum log x}{n}\right)$$

## G.M in Discrete series

$$G.M = Antilog\left(\frac{\sum f log x}{n}\right)$$

G.M in continuous series

$$G.m = Antilog\left(\frac{\sum f log x}{n}\right)$$

x = midpoint of x

1) Find Geometric mean of the following

57.5, 87.75, 53.5, 73.5, 81.75

**Answer:**

| X | logx |
|---|---|
| 57.5 | 1.7597 |
| 87.75 | 1.9432 |
| 53.5 | 1.7284 |
| 73.5 | 1.8663 |
| 81.75 | 1.9125 |
| | 9.2101 ==== |

G.M. = $Antilog\left(\frac{\sum logx}{n}\right)$

$= Antilog\left(\frac{9.2101}{5}\right)$

$= Antilog$ (1.84202)

= 69.51

====

2) Find the G.M 2, 4, 8, 12, 16, 24

| X | logx |
|---|---|
| 2 | 0.3010 |
| 4 | 0.6021 |
| 8 | 0.9031 |
| 12 | 1.0792 |
| 16 | 1.2041 |
| 24 | 1.3802 |
| | 5.4697 |

G.M. = $Antilog\left(\frac{\sum logx}{n}\right)$

$= Antilog\left(\frac{5.4697}{6}\right)$

$= Antilog$ (.9116)

= 8.158

====

3) Find G.M from the following data

| | | | | |
|---|---|---|---|---|
| Size : | 5 | 8 | 10 | 12 |
| Frequency: | 2 | 3 | 4 | 1 |

Ans:

| X | *f* | *logX* | *f logx* |
|---|---|---|---|
| *5* | *2* | *.6990* | *1.3980* |
| *8* | *3* | *.9031* | *2.7093* |
| *10* | *4* | *1.0000* | *4.0000* |
| *12* | *1* | *1.0792* | *1.0792* |
| | *10* | | *9.1865* |

G.M. = $Antilog\left(\frac{\sum logx}{N}\right)$

$= Antilog\left(\frac{9.1865}{10}\right)$

$= Antilog$ (.91865)

= 8.292

====

4) Calculate G.M.

| Daily Income ( ₹ ) | 0-20 | 20-40 | 40-60 | 60-80 | 80-100 |
|---|---|---|---|---|---|
| No. of workers | 5 | 7 | 12 | 8 | 4 |

**Answer :**

| X | *f* | *x(* | *logx* | *f logx* |
|---|---|---|---|---|
| 0-20 | 5 | 10 | 1.0000 | 5.0000 |
| 20-40 | 7 | 20 | 1.4771 | 10.3397 |
| 40-60 | 12 | 30 | 1.6990 | 20.3880 |
| 60-80 | 8 | 40 | 1.8451 | 14.7608 |
| 80-100 | 4 | 50 | 1.9542 | 7.8168 |
| | 36 | | | 58.3053 |

G.M. = $Antilog\left(\frac{\sum logx}{N}\right)$

$= Antilog\left(\frac{58.3053}{36}\right)$

$= Antilog$ 1.6195916

= 41.65
====

## Harmonic Mean

Harmonic mean is defined as the reciprocal of the mean of the reciprocals of those values. It applied in averaging rates, times etc.

$$H.M = \frac{n}{\sum\frac{1}{X}}$$

H.M in Discrete series

$$H.M = \frac{N}{\sum f\left(\frac{1}{x}\right)}$$

H.M in continuous series

$$H.M = \frac{N}{\sum f\left(\frac{1}{x}\right)}$$

$x$ = midpoint of x

1) Calculate H.M. from the following

1)Find the H.M.

2, 3, 4, 5

**Answer:**

| x | $\frac{1}{x}$ |
|---|---|
| 2 | 0.5 |
| 3 | 0.33 |
| 4 | 0.25 |
| 5 | 0.20 |
| | 1.28 |

$$H.M. = \left(\frac{n}{\Sigma\frac{1}{x}}\right)$$

$$= \frac{4}{1.28}$$

$$= 3.125$$

====

2) Find the H.M.

| | | | | |
|---|---|---|---|---|
| Size | 6 | 10 | 14 | 18 |
| F | 20 | 40 | 30 | 10 |

Answer :

| *Size* | *f* | $\frac{1}{x}$ | $f(^1/_X)$ |
|---|---|---|---|
| 6 | 20 | 0.1667 | 3.334 |
| 10 | 40 | 0.1000 | 4.000 |
| 14 | 30 | 0.0714 | 2.142 |
| 18 | 10 | 0.0556 | 0.556 |
| | 100 | | 10.032 |

$$H.M = \frac{N}{\Sigma f(^1/_x)} = \frac{100}{10.032} = 9.97$$

===

3) From the following data, calculate the value of HM?

| Income ( ₹) | No. of persons |
|---|---|
| 10 – 20 | 4 |
| 20 – 30 | 6 |
| 30 – 40 | 10 |
| 40 – 50 | 7 |
| 50 – 60 | 3 |

*Ans:*

| Income ( ₹) | f | x in m | $\frac{1}{x}$ | $f(\frac{1}{x})$ |
|---|---|---|---|---|
| 10 – 20 | 4 | 15 | 0.667 | 0.2666 |
| 20 – 30 | 6 | 25 | 0.0400 | 0.2400 |
| 30 – 40 | 10 | 35 | 0.0286 | 0.2857 |
| 40 – 50 | 7 | 45 | 0.0222 | 0.1556 |
| 50 – 60 | 3 | 55 | 0.0182 | 0.0545 |
| | 30 | | | 1.0023 |

$$HM = \frac{N}{\sum f(\frac{1}{x})} = \frac{30}{1.0023} = 29.93$$

=========

**<u>MEASURES OF DISPERSION OR VARIABILITY</u>**

Dispersion means a measure of the degree of deviation of data from the central value.

Measures of Dispersion are classified into (1) Absolute Measures
(2) Relative Measures.

Absolute Measures of dispersion are expressed in the same units in which data are collected. They measure variability of series. Various absolute measures are:

(i) Range
(ii) Quartile Deviation
(iii) Mean Deviation
(iv) Standard Deviation

Relative measure is also called coefficient of dispersion. They are useful for comparing two series for their variability. Various relative measures are:

(i) Coefficient Range

(ii) Coefficient of Quartile Deviation

(iii) Coefficient of Mean Deviation

(iv) Coefficient of Variation

**RANGE**

The range of any series is the difference between the highest and the lowest values in the series.

Range = H – L

H = Highest variable

L = Lowest variable

Coefficient of Range = $\frac{H-L}{H+L}$

1) Find the Range and Coefficient of Range.

75, 29, 96, 15,7,8,11, 7, 49

*Ans:*

Range = H – L

= 96 – 74 = 92

======

Coefficient of Range = $\frac{H-L}{H+L}$ = $\frac{96-4}{96+4}$ = $\frac{92}{100}$ = 0.92

=======

2) Find Range and Coefficient of Range.

| | | | | | | |
|---|---|---|---|---|---|---|
| Wages | 5 | 10 | 15 | 20 | 25 | 30 |
| No. of employees | 2 | 5 | 6 | 7 | 4 | 6 |

*Ans:*

Range = H – L

= 30 – 5 = 25

======

Coefficient of Range = $\frac{H-L}{H+L}$ = $\frac{30-5}{30+5}$ = $\frac{25}{35}$ = 0.71

==========

3) Find out Range and Coefficient of Range.

| Marks | 20 – 29 | 30 – 39 | 40 – 49 | 50 – 59 | 60 – 69 |
|---|---|---|---|---|---|
| No. of Students | 8 | 12 | 20 | 7 | 3 |

*Ans:*

| Marks | f |
|---|---|
| 19.5 – 29.5 | 8 |
| 29.5 – 39.5 | 12 |
| 39.5 – 49.5 | 20 |
| 49.5 – 59.5 | 7 |
| 59.5 – 69.5 | 3 |

Range = H – L

= 69.5 – 19.5 = 50

====

Coefficient of Range = $\frac{H-L}{H+L}$ = $\frac{69.5-19.5}{69.5+19.5}$ = $\frac{50}{89}$ = 0.56

=====

## QUARTILE DEVIATION

Quartile Deviation is defined as the half distance between the third and first quartiles.

Quartile Deviation = $\frac{Q_3-Q_1}{2}$

Coefficient of Quartile Deviation = $\frac{Q_3-Q_1}{Q3+Q1}$

## Quartile Deviation in Individual Series

Quartile Deviation = $\frac{Q_3-Q_1}{2}$

$Q_1$ = size of $\frac{n+1}{4}$ th Item

$Q_3$ = size of 3 $\left(\frac{n+1}{4}\right)$ th item

**Quartile Deviation in Discrete Series**

Quartile Deviation = $\frac{Q_3 - Q_1}{2}$

$Q_1$ = size of $\frac{N+1}{4}$ th Item

$Q_3$ = size of 3 $\left(\frac{N+1}{4}\right)$ th item

**Quartile Deviation in Continuous Series**

Quartile Deviation = $\frac{Q_3 - Q_1}{2}$

$Q_1$ = size of $\frac{N}{4}$ th Item

$Q_3$ = size of 3 $\left(\frac{N}{4}\right)$ th item

Then, $Q_1 = L_1 + \frac{\frac{N}{4} - c.f}{f} \times c$

$Q_3 = L_1 + \frac{3\left(\frac{N}{4}\right) - cf}{f} \times c$

4) Calculate Quartile Deviation from the following:

25, 15, 30, 45, 40, 20, 50

Also find coefficient of quartile deviation.

*Ans:* Arrange the series, then

15, 20, 25, 30, 40, 45, 50

$Q_1 = \frac{n+1}{4}$ th Item = $\frac{8}{4}$ = 2nd Item

= 20

$Q_3 = 3\left(\frac{n+1}{4}\right)$ th item = 3 x 2 = 6th Item

= 45

Quartile Deviation = $\frac{Q_3 - Q_1}{2} = \frac{45-20}{2}$ = 12.5

Coefficient of Quartile Deviation = $\frac{Q_3 - Q_1}{Q3+Q1}$

$= \frac{25}{45+20} = \frac{20}{65} = 0.385$

=======

2) Find Quartile Deviation and Coefficient of Quartile Deviation.

23, 25, 8, 10, 9, 29, 45, 85, 10, 16

*Ans:* Arrange the series, then

8, 9, 10, 10, 16, 23, 25, 29, 45, 85

$Q_1$ = size of $\frac{n+1}{4}$ th Item = $\frac{10+1}{4}$ th Item = 2.75$^{th}$ Item

ie., 2$^{nd}$ Item + .75 (3$^{rd}$ Item – 2$^{nd}$ Item)

= 9 + .75 (10 – 9)

= 9 + .75 X 1 = 9.75

$Q_3$ = size of 3 $\left(\frac{n+1}{4}\right)$ th item

= 3 x 2.75 = 8.25$^{th}$ Item

i.e. 8$^{th}$ item + .25(9$^{th}$ Item – 8$^{th}$ Item)

= 29 + .25 (45 – 29)

= 29 + .25 x 16

= 29 + 4 = 33

Quartile Deviation = $\frac{Q_3-Q_1}{2}$ = $\frac{33-9.75}{2}$ = 11.625

==========

Coefficient of Quartile Deviation = $\frac{Q_3-Q_1}{Q3+Q1}$

= $\frac{33-9.75}{33+9.75}$ = 0.54

========

3) Find the value of Quartile Deviation and coefficient of Quartile Deviation?

| Marks | 25 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|
| No. of Students | 4 | 7 | 12 | 8 | 9 | 15 | 7 | 3 |

Ans:

| x | f | c.f. | |
|---|---|---|---|
| 25 | 4 | 4 | |
| 30 | 7 | 11 | |
| 40 | 12 | 23 | $Q_1$ |
| 50 | 8 | 31 | |
| 60 | 9 | 40 | |
| 70 | 15 | 55 | $Q_3$ |
| 80 | 7 | 62 | |
| 90 | 3 | 65 | |
| | 65 | | |

$Q_1 = \frac{n+1}{4}$ th Item = $\frac{65+1}{4}$ th Item = 16.5th Item

$Q_3 = 3\left(\frac{n+1}{4}\right)$ th item = 3 x 16.5 = 49.5th Item

$Q_1 = 45$

$Q_3 = 70$

Quartile Deviation = $\frac{Q_3-Q_1}{2} = \frac{70-40}{2}$ = 15 marks

===========

Coefficient of Quartile Deviation = $\frac{Q_3-Q_1}{Q3+Q1}$

$= \frac{70-40}{70+40} = 0.27$

=======

4) Compute Quartile Deviation and coefficient of Quartile Deviation?

| x | 0 - 10 | 10 - 20 | 20 - 30 | 30 - 40 | 40 - 50 | 50 - 60 |
|---|---|---|---|---|---|---|
| f | 5 | 12 | 15 | 9 | 10 | 3 |

Ans:

| x | f | c.f. |
|---|---|---|
| 0 – 10 | 5 | 5 |
| 10 – 20 | 12 | 17 |
| 20 – 30 | 15 | 32 |
| 30 – 40 | 9 | 41 |
| 40 – 50 | 10 | 51 |
| 50 - 60 | 3 | 54 |
| | 54 | |

$Q_1$ =size of $\frac{N}{4}$ th Item = $\frac{54}{4}$ th Item = 13.5th Item

Which lies in 10 – 20 , then

$$Q_1 = L_1 + \frac{\frac{N}{4} - c.f}{f} \text{ x c}$$

$$= 10 + \frac{13.5-5}{12} \text{ x } 10$$

$$= 10 + \frac{8.5}{12} \text{ x } 10$$

$$= 10 + \frac{85}{12} = 17.08$$

=========

$Q_3 = 3\left(\frac{N}{4}\right)$ th item

= 3 x 13.5 = 40.5th Item

Which lies in 30 – 40, then

$$Q_3 = L_1 + \frac{3\left(\frac{N}{4}\right) - cf}{f} \text{ x c}$$

$$= 30 + \frac{40.5-32}{9} \text{ x } 10$$

$$= 30 + \frac{8.5}{9} \text{ x } 10$$

$$= 30 + \frac{85}{9} = 39.44$$

=========

Quartile Deviation = $\frac{Q_3-Q_1}{2}$ = $\frac{39.44-17.08}{2}$ = $\frac{22.36}{2}$ = 11.18 marks

=============

Coefficient of Quartile Deviation = $\frac{Q_3-Q_1}{Q3+Q1}$

= $\frac{39.44-17.08}{39.44+17.08}$

= $\frac{22.36}{56.52}$ = 0.396

==========

**MEAN DEVIATION**

Mean Deviation is defined as the arithmetic mean of deviations of all the values in a series from their average. The average may be mean, median or mode.

Mean Deviation = $\frac{\sum|d|}{n}$

Where $|d|$ = deviation from an average without sign

**Mean Deviation in Individual Series**

Mean Deviation = $\frac{\sum|d|}{n}$

Coefficient of Mean Deviation = $\frac{Mean\ Deviation}{Average}$

Average = Mean, Median or Mode from which the deviation is taken

**Mean Deviation in Discrete Series**

Mean Deviation = $\frac{\sum f|d|}{N}$

Coefficient of Mean Deviation = $\frac{Mean\ Deviation}{Average}$

**Mean Deviation in Continuous Series**

Mean Deviation = $\frac{\sum f|d|}{N}$

1) Calculate Mean Deviation from the following.

14, 15, 23, 20, 10, 30, 19, 18, 16, 25, 12

*Ans:*

Arrange the data

10, 12, 14, 15, 16, 18, 19, 20, 23, 25, 30

Median = size of $\frac{11+1}{2}$ item

= $6^{th}$ Item = 18

| X | $\|d\|$ ie. X – median |
|---|---|
| 14 | 4 |
| 15 | 3 |
| 23 | 5 |
| 20 | 2 |
| 10 | 8 |
| 30 | 12 |
| 19 | 1 |
| 18 | 0 |
| 16 | 2 |
| 25 | 7 |
| 12 | 6 |
| | 50 |

Mean Deviation = $\frac{\sum|d|}{n}$ = $\frac{50}{11}$ = 4.54 marks

============

2) Calculate Mean Deviation from the following data:

| Size of item | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| Freequency | 3 | 6 | 9 | 13 | 8 | 5 | 4 |

*Ans:*

| Size | f | c.f | $\|d\|$ | f $\|d\|$ |
|---|---|---|---|---|
| 6 | 3 | 3 | 3 | 9 |
| 7 | 6 | 9 | 2 | 12 |
| 8 | 9 | 18 | 1 | 9 |
| 9 | 13 | 31 | 0 | 0 |
| 10 | 8 | 39 | 1 | 8 |
| 11 | 5 | 44 | 2 | 10 |
| 12 | 4 | 48 | 3 | 12 |
| | 48 | | | 60 |

Median = $\frac{48+1}{2}^{th}$ item = 24.5

Median = 9

= 18

Mean Deviation = $\frac{\sum f|d|}{N}$ = $\frac{60}{48}$ = 1.25

============

3) Calculate the Mean Deviation from the following data:

| Marks | 0 - 10 | 10 - 20 | 20 - 30 | 30 - 40 | 40 - 50 | 50 - 60 | 60 - 70 | 70 - 80 |
|---|---|---|---|---|---|---|---|---|
| Freequency | 18 | 16 | 15 | 12 | 10 | 5 | 2 | 2 |

Ans:

| x | f | m | c.f. | $\|d\|$ ie. X – median | f$\|d\|$ |
|---|---|---|---|---|---|
| 0 – 10 | 18 | 5 | 18 | 19 | 342 |
| 10 – 20 | 16 | 15 | 34 | 9 | 144 |
| 20 – 30 | 15 | 25 | 49 | 1 | 15 |
| 30 – 40 | 12 | 35 | 61 | 11 | 132 |
| 40 – 50 | 10 | 45 | 71 | 21 | 210 |
| 50 - 60 | 5 | 55 | 76 | 31 | 155 |
| 60 – 70 | 2 | 65 | 78 | 41 | 82 |
| 70 - 80 | 2 | 75 | 80 | 51 | 102 |
| | 80 | | | | 1182 |

Median = $\frac{N}{2}$ th Item

= $\frac{80}{2}$ th Item = $40^{th}$ Item

Which lies on 20 – 30

Median = 20 + $\frac{40-34}{15}$ x 10

= 20 + $\frac{6}{15}$ x 10

= 24

Mean Deviation = $\frac{\sum f|d|}{N}$ = $\frac{1182}{80}$ = 14.775

============

**STANDARD DEVIATION**

Standard Deviation is defined as the square root of the mean of the squares of the deviations of individual items from their arithmetic mean. It is denoted by σ (sigma).

$$\sigma = \frac{\sqrt{\sum(x-\bar{x})^2}}{2}$$

**Standard Deviation in Individual Series**

$$\sigma = \frac{\sqrt{\sum(x-\bar{x})^2}}{n} \quad \text{or} \quad \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

Coefficient of variation = $\frac{\sigma}{\bar{x}}$ x 100

**Standard Deviation in Discrete Series**

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

Shortcut method:

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

d = x - A

**Standard Deviation in Continuous Series**

(i) Direct Method:

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

x = mid point of X

(ii) Shortcut method:

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

d = m – A or x – A

(iii) Step Deviation method:

$$\sigma = \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \text{ X C}$$

d' = $\frac{d}{c}$, c = class interval.

**VARIANCE**

Variance is defined as the mean of the squares of the deviations of all the values in the series from their mean. It is the sqare root of the Standard Deviation.

Variance = $\sigma^2$

1) Compute S.D

4, 8, 10, 12, 15, 9, 7, 7

Ans:

| X | X² |
|---|---|
| 4 | 16 |
| 8 | 64 |
| 10 | 100 |
| 12 | 144 |
| 15 | 225 |
| 8 | 81 |
| 7 | 49 |
| 7 | 49 |
| **72** | **728** |

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{728}{8} - \left(\frac{72}{8}\right)^2}$$

$$= \sqrt{91 - 9^2}$$

$$\sigma = \sqrt{91 - 81} = \sqrt{10}$$

$$= 3.16$$

===

2) Find the S.D and C.V

10, 12, 80, 70, 60, 100, 0, 4

Ans:

| X | $X^2$ |
|---|---|
| 10 | 100 |
| 12 | 144 |
| 80 | 6400 |
| 70 | 4900 |
| 60 | 3600 |
| 100 | 10000 |
| 0 | 0 |
| 4 | 16 |
| **336** | **25160** |

$$\sigma = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$$

$$= \sqrt{\frac{25160}{8} - \left(\frac{336}{8}\right)^2}$$

$$= \sqrt{3145 - (42)^2}$$

$$= \sqrt{3145 - 1764} = \sqrt{1381}$$

$$= 37.16$$

====

| | |
|---|---|
| C.V. | $= \frac{\sigma}{\overline{X}} \times 100$ |
| $\overline{X}$ | $= \frac{336}{8} = 42$ |
| C.V | $= \frac{37.16}{42} \times 100 = 88.48$ <br> ==== |

3) Find out S.D

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Production in tones : | 50 | 100 | 125 | 150 | 200 | 250 | 300 |
| No. of factories: | 2 | 5 | 7 | 12 | 9 | 5 | 3 |

Ans:

| X | $f$ | d(x-A) | $d^1$ | $d^{1^2}$ | $fd^1$ | $f^{d^{1^2}}$ |
|---|---|---|---|---|---|---|
| 50 | 2 | -100 | -4 | 16 | -8 | 32 |
| 100 | 5 | -50 | -2 | 4 | -10 | 20 |
| 125 | 7 | -25 | -1 | 1 | -7 | 7 |
| 150 | 12 | 0 | 0 | 0 | 0 | 0 |
| 200 | 9 | 50 | 2 | 4 | 18 | 36 |
| 250 | 5 | 100 | 4 | 16 | 20 | 80 |
| 300 | 3 | 150 | 6 | 36 | 18 | 108 |
| | **43** | | | | **31** | **283** |

A = 150

$d^1 = \frac{d}{25}$

$$\sigma = \sqrt{\frac{\sum fd^{1^2}}{N} - \left(\frac{\sum fd^1}{N}\right)^2} \times C$$

$$= \sqrt{\frac{283}{43} - \left(\frac{31}{43}\right)^2} \times 25$$

$$= \sqrt{6.58 - 0.52} \times 25$$

$$= \sqrt{6.06} \times 25 = 2.46 \times 25$$

$$= 61.5$$

====

4) Compute the S.D from the following

| Expenditure (Rs): | 100-200 | 200-300 | 300-400 | 400-500 | 500-600 |
|---|---|---|---|---|---|
| No. of families | 30 | 20 | 40 | 5 | 10 |

Ans:

| X | $f$ | m | d( | $d^1$ | $d^{1^2}$ | $fd^1$ | $fd^{1^2}$ |
|---|---|---|---|---|---|---|---|
| 100-200 | 30 | 150 | -200 | -2 | 4 | -60 | 120 |
| 200-300 | 20 | 250 | -100 | -1 | 1 | -20 | 20 |
| 300-400 | 40 | 350 | 0 | 0 | 0 | 0 | 0 |
| 400-500 | 5 | 450 | 100 | 1 | 1 | 5 | 5 |
| 500-600 | 10 | 550 | 200 | 2 | 4 | 20 | 40 |
| | **105** | | | | | **-55** | **185** |

d = m-A

$d^1$ = d/100

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \times C$$

$$= \sqrt{\frac{185}{105} - \left(\frac{-55}{105}\right)^2} \times 100$$

= 122
====

5) The scores of the batsmen A and B the six innings during a certain match are as follows.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Batsman A: | 10 | 12 | 80 | 70 | 60 | 100 | 0 | 4 |
| Batsman B: | 8 | 9 | 7 | 10 | 5 | 9 | 10 | 8 |

(i) Find which of the two batsman is more consistant in scoring.

(ii) Find who is more efficient batchman.

Ans:

| **Batsman A** | | **Batsman B** | |
|---|---|---|---|
| **X** | **$X^2$** | **X** | **$X^2$** |
| 10 | 100 | 8 | 64 |
| 12 | 144 | 9 | 81 |
| 80 | 6400 | 7 | 49 |
| 70 | 4900 | 10 | 100 |
| 60 | 3600 | 5 | 25 |
| 100 | 10000 | 9 | 81 |
| 0 | 0 | 10 | 100 |
| 4 | 16 | 8 | 64 |
| **336** | **25160** | **66** | **564** |

(i) For finding consistant, C.V is calculated

$$C.V = \frac{\sigma}{\overline{X}} \times 100$$

<u>Batsman A</u>

$$\overline{X} = \frac{336}{8} = 42$$

<u>Batsman B</u>

$$\overline{X} = \frac{66}{8} = 8.25$$

$$\sigma = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$$

Batsman A:

$$\sigma = \sqrt{\frac{25160}{8} - \left(\frac{236}{8}\right)^2}$$

$$= 37.16$$

====

$$C.V = \frac{37.16}{42} \times 100$$

$$= 88.48$$

====

Batsman B:

$$\sigma = \sqrt{\frac{564}{8} - \left(\frac{66}{8}\right)^2}$$

$$= 1.562$$

====

$$\sigma = \sqrt{\frac{564}{8} - \left(\frac{66}{8}\right)^2}$$

$$= 18.93$$

====

B is more consistent since C.V. is less.

(ii) For finding more efficient, average is taken

A = 42 B = 8.25

Batsman A is more consistent since he has greater average.

Merits of S.D

1. S.D. is based on all the values of a series.
2. It is rigidly defined
3. It is capable of further mathematical treatment.
4. It is not much affected by sampling fluctuations.

Demerits

1. It is difficult to calculate.
2. Signs of the deviations are not ignored.

Measures of skewness

Skewness means lack of symmetry when a frequency distribution is not symmetrical, it is said t be asymmetrical or skewed. In the case of a skewed distribution, the mean, median and mode are not equal. Similarly for a skewed distribution $Q_1$ and $Q_3$ will not be equidistant from median. It is an asymmetrical distribution. It has a long tail on one side and a start tail on the other side.

A distribution is said to be skewed when:

(1) Mean, media and mode are not equal.

(2) $Q_1$ and $Q_3$ are not equidistant from median.

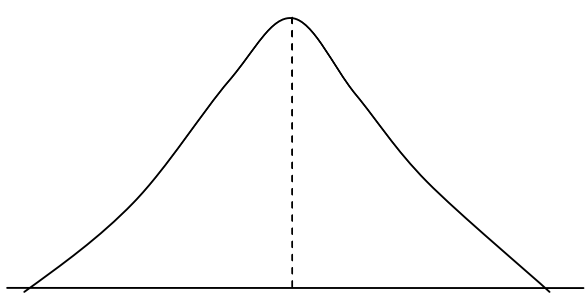
(3) Frequencies on either side of mode are not equal.

(4) The frequency curve has longer tail on the left side or on the right side.

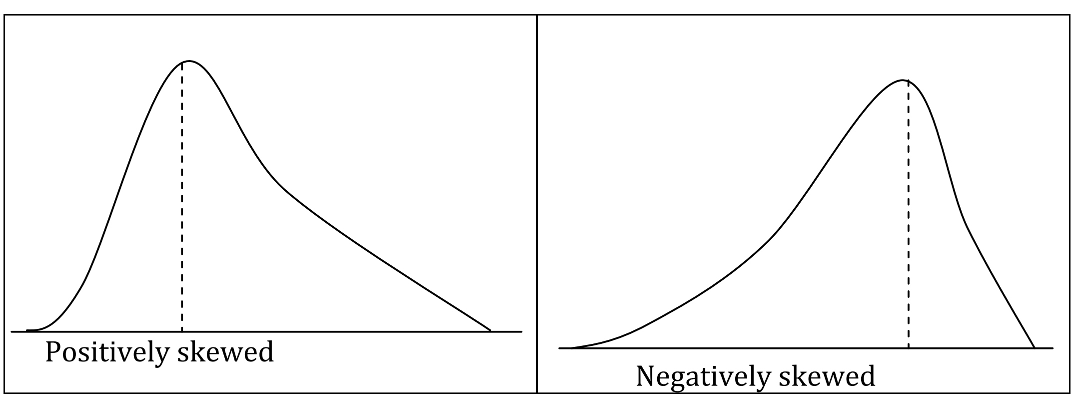
**Positive and Negative skewness**

Skewnwss may be either positive or negative. Skewness is said to be positive when the mean is greater than the median and median is greater than mode. More than half area falls to right side of the highest ordinate.

Swewness is said to be negative when the mean is less than median and the median is less than mode. In this case curve is skewed to the left more than half the area falls to the left of the highest ordinate.



Symmetric Distribution



Positively skewed

Negatively skewed

**Measures of skewness**

1) Karl Pearson's measure of skewness

$$\text{Skewness} = \frac{Mean - Median}{\sigma}$$

2) Bowley's measure of skewness

$$\text{Skewness} = \frac{Q_3 + Q_1 - 2Median}{Q_3 - Q_1}$$

3) Kelley's measure of skewness

$$\text{Skewness} = \frac{P_{90} + P_{10} - 2Median}{P_{90} - P_{10}}$$

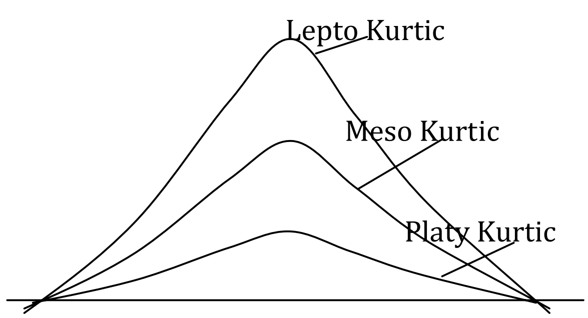
4) Measure of skewness Based on Moments

$$\text{Skewness} = \frac{M_3}{\sqrt{{M_2}^3}}$$

**Kurtosis**

Kurtosis is a measure of peakdness. It refers a distribution which is relatively fetker than the normal curve.

When a frequency curve is more peaked than the normal curve, it is called lepto kurtic and when it is more flat topped than the normal curve it is called platy kurtic. When a curve is neither peaked nor plat topped, it is called meso kurtic normal.




**Lorenz Curve**

Lorenz curve is a graphical method of studying dispersion. It is used in business to study the disparities of the distribution of wages, sales, production etc. In Economics it is useful to measure inequalities in the distribution of income.

It is a graph down to a frequency distribution. While drawing the graph, cumulative percentage values of frequencies on X axis and cumulative percentage values of the variable on Y axis.

**Index Numbers**

Index numbers is a statistical device for measuring the changes in group of related variables over a period of time.

**Uses or Importance of index numbers.**

1. Index numbers measure trend values.
2. Index numbers facilitate for policy decisions.
3. Index numbers help in comparing the standard of living.
4. It measures changes in price level.
5. Index numbers are economic barometers. The condition of the economy of a country to be known through construction of index numbers for different periods with regard to employment, literacy, agriculture industry, economics etc. Hence it can be termed as economic barometers.

**Limitations**

1. Index numbers are only approximate indicator.
2. All index numbers are not good for all purposes.
3. Index numbers are liable to be unissued.
4. Index numbers are specilised average and limitations of average also applicable to index numbers.

**Problems or Difficulties in the construction of index numbers**

1. Purpose of the index.
2. Selection of the lease period.
3. Selection of items.
4. Selection of an average
5. Selection of weights
6. Selection of appropriate source of data
7. Selection of suitable formula.

**Methods of constructing index numbers**

1. Unweighted index numbers.
2. Weighted index numbers.

**Unweighted or Simple index numbers**

Simple index numbers are those index numbers in which all items are treated as equally. Simple aggregate and simple average price relatives are the unweighted index numbers.

**(1) Simple Aggregate method**

$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100$

$P_{01}$ = index number

$P_1$ = Price for the current year

$P_1$ = Price for the base year.

**(2) Simple Average Price Relative Method**

Price index = $\frac{\sum I}{n}$

I = $\frac{P_1}{P_0}$ x 100, each items can be calculated.

**Weighted index numbers**

In this method quantity consumed is also taken into account.

Such index are

1. Weighted aggregate method
2. Weighted Average of price relatives

**Weighted aggregate method**

This method is based on the weight of the prices of the selected commodities.

Following are the commonly used methods:

1. Laspeyre's Method
2. Paasche's Method
3. Bowley-Dorbish Method
4. Fishers ideal method
5. Kelly's Methods

**Laspeyre's Method**

$P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$

$p_1$ = Price of the current year

$q_0$ = Quantity of the base year

$p_0$ = Price of the base year

**Paasche's Method**

$P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1}$ x 100

$q_1$ = Quantity of the current year

**Fishers Ideal Method**

$P_{01} = \sqrt{L \times P}$ x 100

L = Laspeyres method

P = Paasche's Method

$P_{01} = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}}$ x 100

**Bowley-Doribish Method**

$P_{01} = \frac{L+P}{2}$

**Kelly's Method**

$P_{01} = \frac{\sum p_1 q}{\sum p_0 q}$ x 100

$q = \frac{q_0+q_1}{2}$

**Weighted Average Price Relative Method**

Index number = $\frac{\sum \text{IV}}{\sum \text{V}}$

V = Weight

$I = \frac{P_1}{P_0}$ x 100

1. Construct index numbers for 2012 on the basis of the price of 2010

| Commodities | Price in 2010 | Price in 2012 |
|---|---|---|
| A | 115 | 130 |
| B | 72 | 89 |
| C | 54 | 75 |
| D | 60 | 72 |
| E | 80 | 105 |

Answer

| Commodities | $P_0$ | $P_1$ |
|---|---|---|
| A | 115 | 130 |
| B | 72 | 89 |
| C | 54 | 75 |
| D | 60 | 72 |
| E | 80 | 105 |
| | **381** | **471** |
| | === | === |

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

$$= \frac{471}{381} \times 100 = 123.62$$

=====

2. Calculate simple index number by average relative method.

| Items | Price of the base year | Price of the current year |
|---|---|---|
| A | 5 | 7 |
| B | 10 | 12 |
| C | 15 | 25 |
| D | 20 | 18 |
| E | 8 | 9 |

Ans:

| Items | $P_0$ | $P_1$ | (ie $\frac{P_1}{P_0} \times 100$) |
|---|---|---|---|
| A | 5 | 7 | 140 |
| B | 10 | 12 | 120 |
| C | 15 | 25 | 166.7 |
| D | 20 | 18 | 90 |
| E | 8 | 9 | 112.5 |
| | | | **629.2** |
| | | | ==== |

Index number = $\frac{\sum I}{n}$

$= \frac{629.6}{5} = 125.84$

=====

3. Following are the data related with the prices and quantities consumed for 2010 and 2012.

| Commodity | 2010 | | 2012 | |
|---|---|---|---|---|
| | Price | Quantity | Price | Quantity |
| Rice | 5 | 15 | 7 | 12 |
| Wheat | 4 | 5 | 6 | 4 |
| Sugar | 7 | 4 | 9 | 3 |
| Tea | 52 | 2 | 55 | 2 |

Construct price index numbers by

(1) Laspeyre's method

(2) Paasche's method

(3) Bowly's – Dorbish method

(4) Fisher's method

**Answer**

| **Commodity** | $p_0$ | $q_0$ | $p_1$ | $q_1$ | $p_1q_0$ | $p_0q_0$ | $p_1q_1$ | $p_0q_1$ |
|---|---|---|---|---|---|---|---|---|
| Rice | 5 | 15 | 7 | 12 | 105 | 75 | 84 | 60 |
| Wheat | 4 | 5 | 6 | 4 | 30 | 20 | 24 | 16 |
| Sugar | 7 | 4 | 9 | 3 | 36 | 28 | 27 | 21 |
| Tea | 12 | 2 | 55 | 2 | 110 | 104 | 110 | 104 |
| | | | | | **281** | **227** | **245** | **201** |

(1) Laspeyre's Method

$p_{0\,1} = \frac{\sum P_1 q_0}{\sum P_0 q_0}$ x 100 $= \frac{281}{227}$ x 100

$= \underline{\underline{123.79}}$

(2) Paascne's method

$p_{0\,1} = \frac{\sum P_1 q_1}{\sum P_0 q_1}$ x 100 $= \frac{245}{201}$ x 100

$= \underline{\underline{121.89}}$

(3) Bowley – Dorbish Method

$p_{0\,1} = \frac{L+P}{2} = \frac{123.79+121.89}{2}$

$= \underline{\underline{122.84}}$

(4) Fisher's Method

$p_{0\,1} = \sqrt{L \text{ x } P}$

$= \sqrt{123.79 \text{ x } 121.89} = 122.84$

4) Calculate index number of price for 2012 on the basis of 2010, from the data given below:

| **Commodities** | **Weight** | **Price 2010** | **Price 2012** |
|---|---|---|---|
| A | 40 | 16 | 20 |
| B | 25 | 40 | 60 |
| C | 5 | 2 | 2 |
| D | 20 | 5 | 6 |
| E | 10 | 2 | 1 |

**<u>Answers</u>**

Price Index Number $= \frac{\sum IV}{\sum V}$

| Commodities | V | $p_0$ | $p_1$ | i.e. $\frac{p_1}{p_0}$ x 100 | IV |
|---|---|---|---|---|---|
| A | 40 | 16 | 20 | 125 | 5000 |
| B | 25 | 40 | 60 | 150 | 3750 |
| C | 5 | 2 | 2 | 100 | 500 |
| D | 20 | 5 | 6 | 120 | 2400 |
| E | 10 | 2 | 1 | 50 | 500 |
| | **100** | | | | **12150** |

Index Number = $\frac{12150}{100}$ = $\underline{\underline{121.5}}$

5) Construct Price Index

| Commodities | Index | Weight |
|---|---|---|
| A | 350 | 5 |
| B | 200 | 2 |
| C | 240 | 3 |
| D | 150 | 1 |
| E | 250 | 2 |

**Answers**

| Commodities | V | I | IV |
|---|---|---|---|
| A | 5 | 350 | 1750 |
| B | 2 | 200 | 400 |
| C | 3 | 240 | 720 |
| D | 1 | 150 | 150 |
| E | 2 | 250 | 500 |
| | **13** | | **3520** |

Index Number = $\frac{\sum IV}{\sum V}$ = $\frac{3520}{13}$ = $\underline{\underline{270.77}}$

**Consumer Price index number of cost of Living index number or Retail Price index number**

Consumer Price index number is also known as copy of Living Index number or Retails Price index number. It is the ration of the monetary expenditures of an individual which secure him the standard of living or total utility in two situations differing only in respect of prices. It represents the average change in prices over a period of time, paid by the consumer for goods and services.

**Steps in the construction of Consumer Price Index**

1. Determination of the class people for whom the index number is to constructed.
2. Selection of Basic period
3. Conducting family budget enquiry
4. Obtaining price quotation
5. Selecting proper weights
6. Selection of suitable methods for constructing index.

**Methods of Constructing Consumer Price Index Number**

(1) Aggregate Expenditure Method

Cost of living Index number = = $\frac{\sum P_1 q_0}{\sum P_0 q_0}$ x 100

(2) Family Budget Method or Average Relative Method

Cost of Living Index = $\frac{\sum IV}{\sum V}$

1) Find cost of Living index

| | Food | Rent | Clothes | Fuel | Miscellaniou |
|---|---|---|---|---|---|
| Expenses on | 35% | 15% | 20% | 10% | 25% |
| Price 2010 | 150 | 30 | 75 | 25 | 40 |
| Price 2012 | 145 | 30 | 65 | 23 | 45 |

What changes the cost of living of 2012 as compare to 2010?

**Answer**

| Expenses | V | $p_0$ | $p_1$ | I | IV |
|---|---|---|---|---|---|
| Food | 35 | 150 | 145 | 96.67 | 3383.45 |
| Rent | 15 | 30 | 30 | 100 | 1500 |
| Cloth | 20 | 75 | 65 | 86.67 | 1733 |
| Fuel | 10 | 25 | 23 | 92 | 920 |
| Misc. | 20 | 40 | 45 | 112.50 | 2250 |
| | | | | | **9786.85** |

Cost of Living Index $= \frac{\sum IV}{\sum V} = \frac{9786.85}{100} = \underline{97.87}$

**Time Series Analysis**

Time series is the arrangement of data according to the time of occurrence. It helps to find our the variations to the value of data due to changes in time.

**Importance**

1. It helps for understanding past behavior
2. It facilitates for forecasting and Planning
3. It facilitates comparison

**Components of Time Series**

1. Secular trend
2. Seasonal Variations
3. Cyclic Variations
4. Irregular Variations

**Secular Trend**

Trend may be defined as the changes over a long period of time. The significance of trend is greater when the period of time is very longer.

Following are the important method of measuring trend.

1. Graphic Method
2. Semi Average Method
3. Moving Average Method
4. Method of Least Squares

2) **Seasonal Variations**:- Seasonal Variations are measured for one calendar year. It is the variations which occur some degree of regularity. For example climate conditions, social customs etc.

3) **Cyclical Variations**:- Cyclical variations are those variation which occur on account of business cycle. They are Prosperity, Dectine, Depression and Recovery.

4) **Irregular fluctuations**:- One changes of variable could not be predicted due to irregular movements. Irregular movements are like changes in technology, war, famines, flood etc.

**Methods of Measuring Trend**

**(1)** **Graphic method:-** It is otherwise known as free hand method. This is the simplest method of measuring trend. Under this method original data are plotted on the graph paper. The plotted points should be joined, we get a curve. A straight line should be drawn through the middle area of the curve. Such line will describe tendency of the data.

**(2)** **Semi Average Method:-** The whole data are divided in to two parts and average of these are to be calculated. The two averages are to be plotted in the graph. The two points plotted should be joined so as to get a straight line. This line is called the ward live.

**(3)** **Method of Moving average:-** Under this method a series of successive average should be calculated from a series of values moving average may be calculated for 3,4,5,6 or 7 years periods.

The moving average can be calculated as follows:

For example 3 years moving average will be $\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}$ and so on.

Five years moving average = $\frac{a+b+c+d+e}{5}, \frac{b+c+d+e+f}{5}$ and so on.

1) Compute 3 yearly moving average from the following data

| Years: | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sales (in 000units) | 55 | 47 | 59 | 151 | 79 | 36 | 45 | 72 | 83 | 89 | 102 |

**Calculation of 3 yearly moving average**

| Year | Sales (in 000 units) | 3 yearly moving total | 3 yearly moving average |
|---|---|---|---|
| 2002 | 55 | ---------- | ----------- |
| 2003 | 47 | ---------- | ---------- |
| 2004 | 59 | 161 | 53.67 |
| 2005 | 151 | 257 | 85.67 |
| 2006 | 79 | 289 | 96.33 |
| 2007 | 36 | 216 | 58.67 |
| 2008 | 45 | 160 | 63.33 |
| 2009 | 72 | 153 | 51 |
| 2010 | 83 | 200 | 66.67 |
| 2011 | 89 | 244 | 81.33 |
| 2012 | 102 | 277 | 91.33 |

2) Calculate 5 yearly moving average

| Years: | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| income (in '000') | 161 | 127 | 152 | 143 | 144 | 167 | 182 | 179 | 152 | 163 | 159 |

**Answers**

| **Year** | **Income (in 000)** | **Five yearly moving total** | **Five yearly moving average** |
|---|---|---|---|
| 2000 | 161 | ---------- | ----------- |
| 2001 | 127 | ---------- | ---------- |
| 2002 | 152 | 727 | 145.4 |
| 2003 | 143 | 733 | 146.6 |
| 2004 | 144 | 788 | 157.6 |
| 2005 | 167 | 815 | 163 |
| 2006 | 182 | 824 | 164.8 |
| 2007 | 179 | 843 | 168.6 |
| 2008 | 152 | 835 | 167 |
| 2009 | 163 | ---------- | ----------- |
| 2010 | 159 | ---------- | ---------- |

**Calculation of moving average for every periods**

1) Calculate the six year moving average

| Years: | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Demand (in tones) | 105 | 120 | 115 | 110 | 100 | 130 | 135 | 160 | 155 | 140 | 145 |

**Answers**

| Year | Demand | 6 years moving total | 6 years moving average | Centered 6 years moving total | Centered 6 year moving average |
|---|---|---|---|---|---|
| 2000 | 105 | ------- | ------- | ------- | ------- |
| 2001 | 120 | -------- | -------- | -------- | -------- |
| 2002 | 115 | --------- | --------- | --------- | --------- |
| 2003 | 110 | 680 | 113.3 | 231.6 | 115.8 |
| 2004 | 100 | 710 | 118.3 | 243.3 | 121.65 |
| 2005 | 130 | 750 | 125 | 256.67 | 128.34 |
| 2006 | 135 | 790 | 131.67 | 268.34 | 134.17 |
| 2007 | 160 | 820 | 136.67 | 280.84 | 140.42 |
| 2008 | 155 | 865 | 144.17 | | |
| 2009 | 140 | | | | |
| 2010 | 145 | | | | |

4) **Method of Least Squares**

This is a popular method of obtaining trend line. The trend line obtained through this method is called line of best fit.

One trend line is represented as

$y = a + bx$

The value of **a** and **b** can be ascertained by solving the following two normal equations.

$\sum y = Na + b\sum x$

$\sum xy = a\sum x + b\sum x^2$

Where **x** represents the time, **y** represents the value, **a** and **b** are constant and **N** represent total number.

When the middle year is taken as the origin, then $\sum x = 0$, then normal equation would be

$\sum xy = Na$

$\sum xy = b\sum x^2$

Hence a = $\frac{\sum xy}{\sum x^2}$

1) Following are the data related with the output of a factory for 7 years

| Years: | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|
| Output (in tones) | 47 | 64 | 77 | 88 | 97 | 109 | 113 |

Calculate the trend values through the method of least squares and also forecast the production 2013 and 2015.

**Answers**

| **Year** **t** | **Production** **y** | **x** **(t – 2009)** | **xy** | $x^2$ |
|---|---|---|---|---|
| 2006 | 47 | -3 | -141 | 9 |
| 2007 | 64 | -2 | -128 | 4 |
| 2008 | 77 | -1 | -77 | 1 |
| 2009 | 88 | 0 | 0 | 0 |
| 2010 | 97 | 1 | 97 | 1 |
| 2011 | 109 | 2 | 218 | 4 |
| 2012 | 113 | 3 | 339 | 9 |
| | **595** | **0** | **308** | **28** |

Here $\sum x = 0$

Then $a = \frac{\sum y}{n} = \frac{595}{7} = 85$

$b = \frac{\sum xy}{\sum x^2} = \frac{308}{28} = 11$

$y = a + bx$

2006 - 85 + 11 x -3 = 52

2007 - 85 + 11 x -2 = 63

2008 - 85 + 11 x -1 = 74

2009 - 85 + 11 x 0 = 85

2010 - 85 + 11 x 1 = 96

2011 - 85 + 11 x 2 = 107

2012 - 85 + 11 x 3 = 118

**Production in 2013**

= 85 x 11 x 4 = 129 tonns

**Production in 2015**

= 85 x 11 x 6 = 151 tonns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